

CMB spectral distortions as solutions to the Boltzmann equations

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Abstract. We propose to re-interpret the cosmic microwave background spectral distortions as solutions to the Boltzmann equation. This approach makes it possible to solve the second order Boltzmann equation explicitly, with the spectral y distortion and the momentum independent second order temperature perturbation, while generation of μ distortion cannot be explained even at second order in this framework. We also extend our method to higher order Boltzmann equations systematically and find new type spectral distortions, assuming that the collision term is linear in the photon distribution functions, namely, in the Thomson scattering limit. As an example, we concretely construct solutions to the cubic order Boltzmann equation and show that the equations are closed with additional three parameters composed of a cubic order temperature perturbation and two cubic order spectral distortions. The linear Sunyaev-Zel'dovich effect whose momentum dependence is different from the usual y distortion is also discussed in the presence of the next leading order Kompaneets terms, and we show that higher order spectral distortions are also generated as a result of the diffusion process in a framework of higher order Boltzmann equations. The method may be applicable to a wider class of problems and has potential to give a general prescription to non-equilibrium physics.

Keywords: CMB spectral distortion, primordial non-Gaussianity

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1 Introduction

The temperature anisotropy analysis of the cosmic microwave background (CMB) has been successful in the past few decades. The standard Λ CDM cosmology explains the existing linear fluctuations very well and shows us a validity of inflationary paradigm. The next issue of modern cosmology can be the determination of the concrete model of the early universe. The CMB temperature bispectrum has been intensely investigated in this context, and the Planck satellite provides us tight constraints on the primordial non-Gaussianity [1]. In consideration of the future observations, the second order Boltzmann theory has been developed [2–6] because secondary effects may mimic the primordial non-Gaussianity [7–17]. The smallness of the principle anisotropies makes it crucial to estimate the secondary anisotropies, and nowadays several numerical codes have been developed to get rid of such contaminations [18–23]. The anisotropies in the y distortions are also discussed in the same context [24, 25].

On the other hand, spectral distortions of the CMB are also investigated for probes of the thermal history of the early universe. One example is the observed thermal Sunyaev-Zel’dovich (tSZ) effect due to hot gas in the intracluster medium of galaxy groups [26–28]. More generally, the spectral distortions are powerful tools to investigate energy injections from several non-trivial processes such as dark matter pair annihilation [29–31], evaporation of the primordial blackholes [32] and dissipations of the primordial fluctuations [33–44]. Recently, anisotropies of the distortions from Silk damping are also discussed for a primordial non-Gaussianity observation [45–53]. The previous analysis is intuitively reasonable but ad hoc since it is based on thermodynamics of each local diffusion patch with a window function introduced by hands. These anisotropies are at second order in the primordial curvature perturbations and can be understood as mode coupling effects in a framework of the second order Boltzmann theory.

In this paper, we give a unified view to the above two issues by explicitly solving the Boltzmann equations for momentum (frequency) dependent temperature perturbations. We notice the momentum dependence of the temperature perturbations which arises in contrast to zeroth and first order. There are infinite number of evolution equations corresponding to the continuous momentum. As pointed out, for example, in [54], we usually integrate the momentum and define the brightness perturbations at second order. This simplification is preferred since the anisotropy experiments do not focus on a spectroscopy; however, we should keep in mind that a lot of information is hidden in the nontrivial momentum dependence. We handle the infinite number of d.o.f. coming from the continuous momentum by replacing them with the infinite number of the parameters describing spectral distortions¹. Then

¹A similar manipulation called *moment expansion* was originally proposed in [55, 56]. The authors also tried to classify the spectral distortions in a systematic way. In their literatures, concrete analyses were not

fortunately, we find that the necessary number of parameters at second order is only two, and the equations are closed. In addition, we point out a possibility to solve the higher order Boltzmann equations systematically by introducing higher order spectral distortions such as linear Sunyaev-Zel'dovich (SZ) effects. As an example, we construct the next leading order spectral distortions in our method. Observing such a tiny higher order spectral distortion can be the future works in the next few decades. On the other hand, the method may be useful as a prescription to non-equilibrium physics or non-linear evolution of the large scale structure (LSS) as fluid dynamics of matters with gravitational interactions.

We organize this paper as follows. In the section 2, we summarize second and third order Boltzmann collision terms for the photon fluid. The ansatz and the individual equations are discussed in the section 3. We solve the equation for the y distortion in the section 4 and confirm the availability of the previous phenomenological estimations. In the section 5, we comment on another spectral distortion called μ distortion. The section 6 is devoted to the extension of the method to higher orders. The appendices provide several definitions and translations from the previous works. We then conclude in the final section.

2 Boltzmann equation

We shall begin with deriving the higher order Boltzmann collision terms to investigate the spectral distortions and higher order temperature anisotropies.

2.1 Set up

Let f and g be the distribution functions for photon and electron, respectively. Ignoring the Pauli blocking factors, the Boltzmann collision term is given as

$$\begin{aligned} \mathcal{C}[f] = & \frac{1}{16\pi} \int \frac{d\mathbf{n}'}{4\pi} d\tilde{p}' \int \frac{d^3\tilde{q}}{(2\pi)^3} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \\ & \times \frac{\tilde{p}'}{\tilde{p}} \frac{1}{E(\tilde{\mathbf{q}})E(\tilde{\mathbf{p}} + \tilde{\mathbf{q}} - \tilde{\mathbf{p}}')} \delta[\tilde{p} + E(\tilde{\mathbf{q}}) - \tilde{p}' - E(\tilde{\mathbf{p}} + \tilde{\mathbf{q}} - \tilde{\mathbf{p}}')] \\ & \times \{g(\tilde{\mathbf{q}}')f(\tilde{\mathbf{p}}')[1 + f(\tilde{\mathbf{p}})] - g(\tilde{\mathbf{q}})f(\tilde{\mathbf{p}})[1 + f(\tilde{\mathbf{p}}')]\}, \end{aligned} \quad (2.1)$$

where tildes imply that they are physical momenta, which are different from the comoving momenta without tildes (e.g. $p(1+z) = \tilde{p}$ with the redshift z). \mathcal{M} is the invariant scattering amplitude and $\mathbf{n}' \equiv \mathbf{p}'/|\mathbf{p}'|$. We shall now expand each part of (2.1) by introducing two parameters: $\epsilon \sim \mathcal{O}(q/m_e)$ and $\eta = \mathcal{O}(|\tilde{\mathbf{p}} - \tilde{\mathbf{p}}'|/m_e)$ ². We consider that ϵ is the order of the

discussed in the presence of second or higher order collision terms, which are crucial for spectral distortions and central topics of this paper.

²We should note that there are two types hierarchies: the inhomogeneity and the electron energy transfer. The former is directly related to the primordial quantum fluctuations, and we usually consider that the magnitude is the order of 10^{-5} at first order if we assume their scale invariance. In this paper, we use the terminologies “first order” and “second order” in terms of the inhomogeneity.

electron bulk velocity $|\mathbf{v}| = \mathcal{O}(10^{-5})$ or thermal motion $\sqrt{T_e/m_e}$. The relation between the two parameters are given as $\epsilon^2 \sim \eta \ll 1$ since the photon number is peaky for $\tilde{p} \sim T_\gamma$, and $T_\gamma \sim T_e$ is manifest if we assume that the system is in kinetic equilibrium³. Let us now show the summary of calculations below.

The Scattering amplitude—. Let us denote the initial state 4-momenta with primes. The scattering amplitude is then written as [57]

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \left[\frac{\tilde{q} \cdot \tilde{p}'}{\tilde{q} \cdot \tilde{p}} + \frac{\tilde{q} \cdot \tilde{p}}{\tilde{q} \cdot \tilde{p}'} - 2m_e^2 \left(\frac{1}{\tilde{q} \cdot \tilde{p}} - \frac{1}{\tilde{q} \cdot \tilde{p}'} \right) + m_e^4 \left(\frac{1}{\tilde{q} \cdot \tilde{p}} - \frac{1}{\tilde{q} \cdot \tilde{p}'} \right)^2 \right], \quad (2.2)$$

where we have used

$$\tilde{p} \cdot \tilde{q} = \tilde{p}' \cdot \tilde{q}', \quad (2.3)$$

$$\tilde{p} \cdot \tilde{q}' = \tilde{p}' \cdot \tilde{q}, \quad (2.4)$$

which are obtained with 4-momentum conservation $\tilde{p} + \tilde{q} = \tilde{p}' + \tilde{q}'$. Let us chose the frame to satisfy $\tilde{q}_\mu = (-\sqrt{m_e^2 + \tilde{\mathbf{q}}^2}, \tilde{\mathbf{q}})$. Using the above, we can expand (2.2) as follows:

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = |\mathcal{M}_0|^2 + |\mathcal{M}_\epsilon|^2 + |\mathcal{M}_{\eta^2}|^2 + |\mathcal{M}_{\epsilon^2}|^2 + |\mathcal{M}_{\epsilon^3}|^2, \quad (2.5)$$

where we have defined

$$\frac{|\mathcal{M}_0|^2}{2e^4} = 1 + \lambda^2, \quad (2.6)$$

$$\frac{|\mathcal{M}_\epsilon|^2}{2e^4} = 2\lambda(\lambda - 1) \left[\frac{\mathbf{n} \cdot \tilde{\mathbf{q}}}{m_e} + \frac{\mathbf{n}' \cdot \tilde{\mathbf{q}}}{m_e} \right], \quad (2.7)$$

$$\frac{|\mathcal{M}_{\eta^2}|^2}{2e^4} = (\lambda - 1)^2 \frac{\tilde{p}^2}{m_e^2}, \quad (2.8)$$

$$\begin{aligned} \frac{|\mathcal{M}_{\epsilon^2}|^2}{2e^4} &= 2\lambda(\lambda - 1) \frac{\tilde{q}^2}{m_e^2} + (\lambda - 1)(3\lambda - 1) \left[\frac{(\mathbf{n}' \cdot \tilde{\mathbf{q}})^2}{m_e^2} + \frac{(\mathbf{n} \cdot \tilde{\mathbf{q}})^2}{m_e^2} \right] \\ &\quad + 2(\lambda - 1)(2\lambda - 1) \frac{(\mathbf{n} \cdot \tilde{\mathbf{q}})(\mathbf{n}' \cdot \tilde{\mathbf{q}})}{m_e^2}. \end{aligned} \quad (2.9)$$

The energy product—. On the other hand, the energy product part can be reduced as follows:

$$\frac{\tilde{p}'}{\tilde{p}} \frac{1}{E(\tilde{\mathbf{q}})E(\tilde{\mathbf{q}}')} = \frac{\tilde{p}'}{\tilde{p}m_e^2} (1 - \mathcal{E}_{\epsilon^2} - \mathcal{E}_{\epsilon\eta} - \mathcal{E}_{\eta^2}), \quad (2.10)$$

³In the hot electron gas in the late universe, $\epsilon^2 \gg \eta$ is possible.

where each part is written as

$$\mathcal{E}_{\epsilon^2} = \frac{\tilde{q}^2}{m_e^2}, \quad (2.11)$$

$$\mathcal{E}_{\epsilon\eta} = \frac{(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}') \cdot \tilde{\mathbf{q}}}{m_e^2}, \quad (2.12)$$

$$\mathcal{E}_{\eta^2} = \frac{(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')^2}{m_e^2}. \quad (2.13)$$

The delta function—.

The delta functions are Taylor expanded around the photon energy difference; namely, we have

$$\begin{aligned} & \delta(\tilde{p} - \tilde{p}' + E(\tilde{\mathbf{q}}) - E(\tilde{\mathbf{q}}')) \\ &= \delta(\tilde{p} - \tilde{p}') + (\mathcal{D}_\epsilon + \mathcal{D}_{\epsilon^3} + \mathcal{D}_\eta) \frac{\partial \delta(\tilde{p} - \tilde{p}')}{\partial \tilde{p}'} \\ &+ \frac{1}{2} (\mathcal{D}_\epsilon + \mathcal{D}_\eta)^2 \frac{\partial^2 \delta(\tilde{p} - \tilde{p}')}{\partial \tilde{p}'^2} + \frac{1}{3!} \mathcal{D}_\epsilon^3 \frac{\partial^3 \delta(\tilde{p} - \tilde{p}')}{\partial \tilde{p}'^3}, \end{aligned} \quad (2.14)$$

where we have introduced

$$\mathcal{D}_\epsilon = \frac{(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}') \cdot \tilde{\mathbf{q}}}{m_e}, \quad (2.15)$$

$$\mathcal{D}_{\epsilon^3} = -\frac{\tilde{q}^2 (\tilde{\mathbf{p}} - \tilde{\mathbf{p}}') \cdot \tilde{\mathbf{q}}}{2m_e^3}, \quad (2.16)$$

$$\mathcal{D}_\eta = \frac{(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')^2}{m_e}. \quad (2.17)$$

Product of distribution functions—. Expanding $g(\tilde{\mathbf{q}}')$ around $\tilde{\mathbf{q}}$, the product of the distribution functions can be written as

$$\begin{aligned} \mathcal{F}(\tilde{\mathbf{p}}', \tilde{\mathbf{p}}) &= f(\tilde{\mathbf{p}}') [1 + f(\tilde{\mathbf{p}})] - \frac{g(\tilde{\mathbf{q}})}{g(\tilde{\mathbf{q}}')} f(\tilde{\mathbf{p}}) [1 + f(\tilde{\mathbf{p}}')] \\ &= \mathcal{F}_1(\tilde{\mathbf{p}}', \tilde{\mathbf{p}}) + (\alpha_{\eta/\epsilon} + \alpha_{\eta^2/\epsilon^2}) \mathcal{F}_3(\tilde{\mathbf{p}}', \tilde{\mathbf{p}}), \end{aligned} \quad (2.18)$$

where \mathcal{F}_1 , \mathcal{F}_3 and α 's are defined as

$$\alpha_{\eta/\epsilon} = -\frac{(\tilde{\mathbf{q}} - m_e \mathbf{v}) \cdot (\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')}{m_e T_e}, \quad (2.19)$$

$$\alpha_{\eta^2/\epsilon^2} = -\frac{(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')^2}{2m_e T_e}, \quad (2.20)$$

$$\mathcal{F}_1(\tilde{\mathbf{p}}', \tilde{\mathbf{p}}) = f(\tilde{\mathbf{p}}') - f(\tilde{\mathbf{p}}), \quad (2.21)$$

$$\mathcal{F}_3(\tilde{\mathbf{p}}', \tilde{\mathbf{p}}) = f(\tilde{\mathbf{p}}') [1 + f(\tilde{\mathbf{p}})]. \quad (2.22)$$

Electron momentum integrals—.

Let us write the $\tilde{\mathbf{q}}$ integral as follows:

$$\langle \cdots \rangle \equiv \int \frac{d^3 \tilde{\mathbf{q}}}{(2\pi)^3} \cdots g(\tilde{\mathbf{q}}). \quad (2.23)$$

Then integrals with the momentum should be recast into

$$\begin{aligned} \langle \tilde{q}_i \rangle &= n_e m_e v_i, \\ \langle \tilde{q}_i \tilde{q}_j \rangle &= n_e m_e T_e \delta_{ij} + n_e m_e^2 v_i v_j, \\ \langle \tilde{q}_i \tilde{q}_j \tilde{q}_k \rangle &= n_e m_e (v_i \langle (\tilde{q}_j - m_e v_j)(\tilde{q}_k - m_e v_k) \rangle + 2\text{perms.}) + n_e m_e^3 v_i v_j v_k \\ &= n_e m_e^2 T_e (v_i \delta_{jk} + 2\text{perms.}) + n_e m_e^3 v_i v_j v_k. \end{aligned} \quad (2.24)$$

2.2 Expansions in terms of ϵ and η

Now we are ready to write the collision terms with respect to ϵ and η :

$$\mathcal{C}[f] = \mathcal{C}_{0,0}[f] + \mathcal{C}_{\epsilon,0}[f] + \mathcal{C}_{\epsilon^2,0}[f] + \mathcal{C}_{\epsilon^3,0}[f] + \mathcal{C}_{0,\eta}[f] + \mathcal{C}_{\epsilon,\eta}[f]. \quad (2.25)$$

We shall now substitute (2.5), (2.10), (2.14) and (2.18) into (2.1). Then, we integrate each momentum by using (2.24) and substitute $\tilde{p} = p(1+z)$. For simplification of the notation, we introduce the following three angular parameters only for this subsection: $\lambda = \mathbf{n} \cdot \mathbf{n}'$, $\mu = \hat{\mathbf{v}} \cdot \mathbf{n}$ and $\mu' = \hat{\mathbf{v}} \cdot \mathbf{n}'$. After tremendous but straight forward calculations, each term is obtained as follows:

$$\mathcal{C}_{0,0}[f] = \frac{3}{4} (\lambda^2 + 1) \mathcal{F}_1(p\mathbf{n}', p\mathbf{n}) \quad (2.26)$$

$$\begin{aligned} \mathcal{C}_{\epsilon,0}[f] = & \frac{3}{4} \left[\{ (4\lambda^2 - 2\lambda + 2) \mu' + (\lambda^2 - 2\lambda - 1) \mu \} \mathcal{F}_1(p\mathbf{n}', p\mathbf{n}) \right. \\ & \left. - (\lambda^2 + 1) (\mu - \mu') p \frac{\partial \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'} \Big|_{p'=p} \right] \end{aligned} \quad (2.27)$$

$$\begin{aligned} \mathcal{C}_{\epsilon^2,0}[f] = & \frac{T_e}{m_e} \left[-\frac{3}{4} (\lambda^3 - \lambda^2 + \lambda - 1) p^2 \frac{\partial^2 \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'^2} \Big|_{p'=p} \right. \\ & - 3 (\lambda^3 - \lambda^2 + \lambda - 1) p \frac{\partial \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'} \Big|_{p'=p} \\ & \left. + \frac{3}{2} (2\lambda^3 - 3\lambda^2 - 2\lambda + 1) \mathcal{F}_1(p\mathbf{n}', p\mathbf{n}) \right] \\ & + v^2 \left[\frac{3}{8} (\lambda^2 + 1) \mu^2 p^2 \frac{\partial^2 \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'^2} \Big|_{p'=p} + \frac{3}{8} (\lambda^2 + 1) \mu'^2 p^2 \frac{\partial^2 \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'^2} \Big|_{p'=p} \right. \\ & - \frac{3}{4} (\lambda^2 + 1) \mu \mu' p^2 \frac{\partial^2 \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'^2} \Big|_{p'=p} - \frac{3}{4} (\lambda^2 - 2\lambda - 1) \mu^2 p \frac{\partial \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'} \Big|_{p'=p} \\ & - \frac{3}{4} (-5\lambda^2 + 2\lambda - 3) \mu'^2 p \frac{\partial \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'} \Big|_{p'=p} - 3 (\lambda^2 + 1) \mu \mu' p \frac{\partial \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'} \Big|_{p'=p} \\ & + \frac{3}{4} [\lambda^2 (\mu^2 - 3) - 2\lambda (\mu^2 - 1) + \mu^2 - 1] \mathcal{F}_1(p\mathbf{n}', p\mathbf{n}) \\ & \left. + \frac{3}{2} (5\lambda^2 - 4\lambda + 2) \mu'^2 \mathcal{F}_1(p\mathbf{n}', p\mathbf{n}) + 3(\lambda - 2) \lambda \mu \mu' \mathcal{F}_1(p\mathbf{n}', p\mathbf{n}) \right] \end{aligned} \quad (2.28)$$

$$\begin{aligned}
& \mathcal{C}_{\epsilon^3,0}[f] \\
&= \frac{T_e}{m_e} v \left[\frac{3}{4} (-11\lambda^3 + 13\lambda^2 - 11\lambda + 9) \mu' p^2 \frac{\partial^2 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^2} \Big|_{p'=p} \right. \\
&+ \frac{3}{2} (2\lambda^3 - \lambda^2 + 2\lambda - 3) \mu p^2 \frac{\partial^2 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^2} \Big|_{p'=p} \\
&+ \frac{3}{4} (\lambda - 1) (\lambda^2 + 1) (\mu - \mu') p^3 \frac{\partial^3 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^3} \Big|_{p'=p} \\
&- \frac{3}{8} (40\lambda^3 - 47\lambda^2 + 56\lambda - 31) \mu' p \frac{\partial \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'} \Big|_{p'=p} \\
&- \frac{3}{8} (16\lambda^3 - 41\lambda^2 + 7) \mu p \frac{\partial \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'} \Big|_{p'=p} + \frac{3}{4} (20\lambda^3 - 46\lambda^2 + 5\lambda + 3) \mu' \mathcal{F}_1(p \mathbf{n}', p \mathbf{n}) \\
&+ \left. \frac{3}{8} (16\lambda^3 - 41\lambda^2 + 34\lambda + 9) \mu \mathcal{F}_1(p \mathbf{n}', p \mathbf{n}) \right] \\
&v^3 \left[\frac{1}{8} (-\lambda^2 - 1) p^3 (\mu - \mu') \mu'^2 \frac{\partial^3 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^3} \Big|_{p'=p} + \frac{1}{4} (\lambda^2 + 1) \mu p^3 (\mu - \mu') \mu' \frac{\partial^3 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^3} \Big|_{p'=p} \right. \\
&+ \frac{1}{8} (-\lambda^2 - 1) \mu^2 p^3 (\mu - \mu') \frac{\partial^3 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^3} \Big|_{p'=p} + \frac{3}{8} (\lambda^2 - 2\lambda - 1) \mu^3 p^2 \frac{\partial^2 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^2} \Big|_{p'=p} \\
&+ \frac{3}{4} (3\lambda^2 - \lambda + 2) p^2 \mu'^3 \frac{\partial^2 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^2} \Big|_{p'=p} + \frac{3}{8} (-11\lambda^2 + 2\lambda - 9) \mu p^2 \mu'^2 \frac{\partial^2 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^2} \Big|_{p'=p} \\
&+ \frac{3}{4} (2\lambda^2 + \lambda + 3) \mu^2 p^2 \mu' \frac{\partial^2 \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'^2} \Big|_{p'=p} \\
&- \frac{3}{8} \mu p (\lambda^2 (2\mu^2 - 7) - 4\lambda (\mu^2 - 1) + 2\mu^2 - 3) \frac{\partial \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'} \Big|_{p'=p} \\
&+ \frac{3}{4} (15\lambda^2 - 10\lambda + 7) p \mu'^3 \frac{\partial \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'} \Big|_{p'=p} - \frac{3}{2} (5\lambda^2 + 4) \mu p \mu'^2 \frac{\partial \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'} \Big|_{p'=p} \\
&- \frac{3}{8} p (\lambda^2 (8\mu^2 + 7) - 4\lambda (4\mu^2 + 1) - 4\mu^2 + 3) \mu' \frac{\partial \mathcal{F}_1(p' \mathbf{n}', p \mathbf{n})}{\partial p'} \Big|_{p'=p} \\
&+ \frac{3}{8} \mu (\lambda^2 (2\mu^2 - 7) + \lambda (14 - 4\mu^2) + 2\mu^2 - 1) \mathcal{F}_1(p \mathbf{n}', p \mathbf{n}) \\
&+ 3 (5\lambda^2 - 5\lambda + 2) \mu'^3 \mathcal{F}_1(p \mathbf{n}', p \mathbf{n}) + \frac{3}{2} (5\lambda^2 - 10\lambda + 2) \mu \mu'^2 \mathcal{F}_1(p \mathbf{n}', p \mathbf{n}) \\
&+ \left. \frac{3}{4} (2\lambda^2 (2\mu^2 - 7) + \lambda (13 - 8\mu^2) + 4\mu^2 - 5) \mu' \mathcal{F}_1(p \mathbf{n}', p \mathbf{n}) \right]
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
\mathcal{C}_{0,\eta}[f] &= \frac{3p(1+z)}{4m_e} (\lambda^3 - \lambda^2 + \lambda - 1) \left[p \frac{\partial \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'} \Big|_{p'=p} - 2p \frac{\partial \mathcal{F}_3(p'\mathbf{n}', p\mathbf{n})}{\partial p'} \Big|_{p'=p} \right] \\
&\quad + \frac{3p(1+z)}{4m_e} (\lambda^3 - \lambda^2 + \lambda - 1) [2\mathcal{F}_1(p\mathbf{n}', p\mathbf{n}) - 4\mathcal{F}_3(p\mathbf{n}', p\mathbf{n})] \\
\mathcal{C}_{\epsilon,\eta}[f] &= v \frac{p(1+z)}{m_e} \left[\frac{3}{4} (\lambda - 1) (-\lambda^2 - 1) (\mu - \mu') \left[p^2 \frac{\partial^2 \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'^2} \Big|_{p'=p} - 2p^2 \frac{\partial^2 \mathcal{F}_3(p'\mathbf{n}', p\mathbf{n})}{\partial p'^2} \Big|_{p'=p} \right] \right. \\
&\quad + \frac{3}{4} (\lambda - 1) \left[4 \{ (-4\lambda^2 + \lambda - 3) \mu' + (\lambda^2 + \lambda + 2) \mu \} p \frac{\partial \mathcal{F}_3(p'\mathbf{n}', p\mathbf{n})}{\partial p'} \Big|_{p'=p} \right. \\
&\quad \left. \left. - 2 \{ (-4\lambda^2 + \lambda - 3) \mu' + (\lambda^2 + \lambda + 2) \mu \} p \frac{\partial \mathcal{F}_1(p'\mathbf{n}', p\mathbf{n})}{\partial p'} \Big|_{p'=p} \right] \right. \\
&\quad + \frac{3}{4} (\lambda - 1) [2 \{ (5\lambda^2 - 2\lambda + 3) \mu' + (\lambda^2 - 2\lambda - 1) \mu \} \mathcal{F}_1(p\mathbf{n}', p\mathbf{n}) \\
&\quad \left. \left. - 4 \{ (5\lambda^2 - 2\lambda + 3) \mu' + (\lambda^2 - 2\lambda - 1) \mu \} \mathcal{F}_3(p\mathbf{n}', p\mathbf{n}) \right] \right]
\end{aligned} \tag{2.31}$$

2.3 Perturbative expansions

Now we integrate the terms with respect to \mathbf{n}' and take the Legendre coefficients. (A.6) is useful for converting $(\mathbf{n} \cdot \mathbf{n}')$ into combinations of $(\hat{\mathbf{v}} \cdot \mathbf{n}')$ and $(\hat{\mathbf{v}} \cdot \mathbf{n})$. In this section, we ignore the vector and tensor sector for simplicity, and rewrite the physical momenta by using the comoving momenta. Here we consider the following perturbative expansion:

$$f = \sum_{n=0}^3 f^{(n)}, \tag{2.32}$$

and replace the bulk velocity as $\mathbf{v} \rightarrow \mathbf{v}^{(1)} + \mathbf{v}^{(2)} + \mathbf{v}^{(3)}$.

Zeroth and first order— We immediately find zeroth and first order collision terms without polarizations at the Thomson scattering limit:

$$(n_e \sigma_T a)^{-1} \mathcal{C}_T^{(0)}[f] = 0, \tag{2.33}$$

$$(n_e \sigma_T a)^{-1} \mathcal{C}_T^{(1)}[f] = f_0^{(1)} - f^{(1)}(\mathbf{p}) - \frac{1}{2} f_2^{(1)} P_2 - p \frac{\partial f^{(0)}}{\partial p} (\mathbf{v} \cdot \mathbf{n}). \tag{2.34}$$

Second order—. The next leading order has two parts. The Thomson terms which do not induce the momentum transfer become

$$\begin{aligned}
& (n_e \sigma_T a)^{-1} \mathcal{C}_T^{(2)}[f] \\
&= f_0^{(2)} - f^{(2)}(\mathbf{p}) - \frac{1}{2} f_2^{(2)} P_2 - p \frac{\partial f^{(0)}}{\partial p} (\mathbf{v}^{(2)} \cdot \mathbf{n}) \\
& \quad v \left[(\hat{\mathbf{v}} \cdot \mathbf{n}) \left(f^{(1)}(\mathbf{p}) - p \frac{\partial f_0^{(1)}}{\partial p} + \frac{1}{5} p \frac{\partial f_2^{(1)}}{\partial p} - f_0^{(1)} + \frac{4}{5} f_2^{(1)} \right) + P_3 \left(\frac{3}{10} p \frac{\partial f_2^{(1)}}{\partial p} - \frac{3}{10} f_2^{(1)} \right) \right. \\
& \quad \left. + i P_2 \left(-\frac{1}{5} p \frac{\partial f_1^{(1)}}{\partial p} + \frac{3}{10} p \frac{\partial f_3^{(1)}}{\partial p} + \frac{1}{5} f_1^{(1)} + \frac{6}{5} f_3^{(1)} \right) - i p \frac{\partial f_1^{(1)}}{\partial p} - 2i f_1^{(1)} \right] \\
& \quad + v^2 \left[P_2 \left(\frac{11}{30} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} + \frac{2}{3} p \frac{\partial f^{(0)}}{\partial p} \right) + \frac{1}{3} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} + \frac{4}{3} p \frac{\partial f^{(0)}}{\partial p} \right] \tag{2.35}
\end{aligned}$$

On the other hand, the Kompaneets terms which have momentum transfer whose magnitude is second order can be written as follows.

$$\begin{aligned}
& (n_e \sigma_T a)^{-1} \mathcal{C}_K^{(0)}[f] = \\
& \frac{T_e}{m_e} \left(p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} + 4p \frac{\partial f^{(0)}}{\partial p} \right) + \frac{p(1+z)}{m_e} \left(2f^{(0)} p \frac{\partial f^{(0)}}{\partial p} + p \frac{\partial f^{(0)}}{\partial p} + 4f^{(0)2} + 4f^{(0)} \right), \tag{2.36}
\end{aligned}$$

These terms are the additional contributions for the averaged part of (2.35).

Cubic order—. The cubic order Thomson terms are obtained as follows:

$$\begin{aligned}
& (n_e \sigma_{\text{Ta}})^{-1} \mathcal{C}_{\text{T}}^{(3)}[f] \\
&= -f^{(3)}(\mathbf{p}) + f_0^{(3)} - \frac{1}{2} P_2 f_2^{(3)} - p \frac{\partial f^{(0)}}{\partial p} (\mathbf{v}^{(3)} \cdot \mathbf{n}) \\
& \quad \left[(\hat{\mathbf{v}} \cdot \mathbf{n}) \left(-\frac{7}{25} p^3 \frac{\partial^3 f^{(0)}}{\partial p^3} - \frac{47}{25} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} - \frac{3}{2} p \frac{\partial f^{(0)}}{\partial p} \right) \right. \\
& \quad \left. + P_3 \left(-\frac{13}{150} p^3 \frac{\partial^3 f^{(0)}}{\partial p^3} - \frac{11}{50} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} \right) \right] v^3 \\
& \quad + \left[\frac{1}{3} p^2 \frac{\partial^2 f_0^{(1)}}{\partial p^2} - \frac{11}{30} p^2 \frac{\partial^2 f_2^{(1)}}{\partial p^2} + \frac{4}{3} p \frac{\partial f_0^{(1)}}{\partial p} - \frac{34}{15} p \frac{\partial f_2^{(1)}}{\partial p} - \frac{12}{5} f_2^{(1)} \right. \\
& \quad \left. + P_4 \left(-\frac{3}{35} p^2 \frac{\partial^2 f_2^{(1)}}{\partial p^2} + \frac{6}{35} p \frac{\partial f_2^{(1)}}{\partial p} - \frac{6}{35} f_2^{(1)} \right) \right. \\
& \quad \left. + (\hat{\mathbf{v}} \cdot \mathbf{n}) \left(\frac{27}{25} i p^2 \frac{\partial^2 f_1^{(1)}}{\partial p^2} - \frac{3}{25} i p^2 \frac{\partial^2 f_3^{(1)}}{\partial p^2} + \frac{108}{25} i p \frac{\partial f_1^{(1)}}{\partial p} - \frac{27}{25} i p \frac{\partial f_3^{(1)}}{\partial p} + \frac{42}{25} i f_1^{(1)} - \frac{48}{25} i f_3^{(1)} \right) \right. \\
& \quad \left. + P_3 \left(\frac{3}{25} i p^2 \frac{\partial^2 f_1^{(1)}}{\partial p^2} - \frac{9}{50} i p^2 \frac{\partial^2 f_3^{(1)}}{\partial p^2} - \frac{3}{25} i p \frac{\partial f_1^{(1)}}{\partial p} - \frac{18}{25} i p \frac{\partial f_3^{(1)}}{\partial p} + \frac{3}{25} i f_1^{(1)} + \frac{18}{25} i f_3^{(1)} \right) \right. \\
& \quad \left. + P_2 \left(\frac{11}{30} p^2 \frac{\partial^2 f_0^{(1)}}{\partial p^2} - \frac{11}{42} p^2 \frac{\partial^2 f_2^{(1)}}{\partial p^2} + \frac{3}{35} p^2 \frac{\partial^2 f_4^{(1)}}{\partial p^2} + \frac{2}{3} p \frac{\partial f_0^{(1)}}{\partial p} \right. \right. \\
& \quad \left. \left. - \frac{22}{21} p \frac{\partial f_2^{(1)}}{\partial p} + \frac{6}{7} p \frac{\partial f_4^{(1)}}{\partial p} + \frac{9 f_2^{(1)}}{7} + \frac{12 f_4^{(1)}}{7} \right) \right] v^2 \\
& \quad + \left[-2i f_1^{(2)} - i p \frac{\partial f_1^{(2)}}{\partial p} + (\hat{\mathbf{v}} \cdot \mathbf{n}) \left(f^{(2)}(\mathbf{p}) - f_0^{(2)} + \frac{4 f_2^{(2)}}{5} - p \frac{\partial f_0^{(2)}}{\partial p} + \frac{1}{5} p \frac{\partial f_2^{(2)}}{\partial p} \right) \right. \\
& \quad \left. + P_3 \left(\frac{3}{10} p \frac{\partial f_2^{(2)}}{\partial p} - \frac{3 f_2^{(2)}}{10} \right) + P_2 \left(\frac{1}{5} i f_1^{(2)} + \frac{6}{5} i f_3^{(2)} - \frac{1}{5} i p \frac{\partial f_1^{(2)}}{\partial p} + \frac{3}{10} i p \frac{\partial f_3^{(2)}}{\partial p} \right) \right] v \\
& \quad + v^{(2)} \left[-2i f_1^{(1)} - i p \frac{\partial f_1^{(1)}}{\partial p} + (\hat{\mathbf{v}} \cdot \mathbf{n}) \left(f^{(1)}(\mathbf{p}) - f_0^{(1)} + \frac{4 f_2^{(1)}}{5} - p \frac{\partial f_0^{(1)}}{\partial p} + \frac{1}{5} p \frac{\partial f_2^{(1)}}{\partial p} \right) \right. \\
& \quad \left. + P_3 \left(\frac{3}{10} p \frac{\partial f_2^{(1)}}{\partial p} - \frac{3 f_2^{(1)}}{10} \right) + P_2 \left(\frac{1}{5} i f_1^{(1)} + \frac{6}{5} i f_3^{(1)} - \frac{1}{5} i p \frac{\partial f_1^{(1)}}{\partial p} + \frac{3}{10} i p \frac{\partial f_3^{(1)}}{\partial p} \right) \right. \\
& \quad \left. + v \left(\frac{2}{3} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} + \frac{8}{3} p \frac{\partial f^{(0)}}{\partial p} + P_2 \left(\frac{11}{15} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} + \frac{4}{3} p \frac{\partial f^{(0)}}{\partial p} \right) \right) \right]. \tag{2.37}
\end{aligned}$$

On the other hand, next leading order Kompaneets terms are given as [37]

$$\begin{aligned}
& (n_e \sigma_{Ta})^{-1} \mathcal{C}_K^{(1)}[f] \\
&= \frac{p(1+z)}{m_e} \left[2p \frac{\partial f^{(0)}}{\partial p} f^{(1)}(\mathbf{p}) + 4f^{(0)} f^{(1)}(\mathbf{p}) + 2f^{(1)}(\mathbf{p}) + 4f^{(0)} f_0^{(1)} + p \frac{\partial f_0^{(1)}}{\partial p} + 2f_0^{(1)} + 2f^{(0)} p \frac{\partial f_0^{(1)}}{\partial p} \right. \\
&+ (\hat{\mathbf{v}} \cdot \mathbf{n}) \left(\frac{24}{5} i f^{(0)} f_1^{(1)} + \frac{12}{5} i f_1^{(1)} + \frac{12i}{5} f^{(0)} p \frac{\partial f_1^{(1)}}{\partial p} + \frac{6i}{5} p \frac{\partial f_1^{(1)}}{\partial p} \right) \\
&+ (\mathbf{v} \cdot \mathbf{n}) \left(-8f^{(0)2} - 8f^{(0)} - \frac{7}{5} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} - \frac{14}{5} f^{(0)} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} - \frac{31}{5} p \frac{\partial f^{(0)}}{\partial p} - \frac{62}{5} f^{(0)} p \frac{\partial f^{(0)}}{\partial p} \right) \\
&+ P_2 \left(-2f^{(0)} f_2^{(1)} - f_2^{(1)} - f^{(0)} p \frac{\partial f_2^{(1)}}{\partial p} - \frac{1}{2} p \frac{\partial f_2^{(1)}}{\partial p} \right) \\
&+ P_3 \left(-\frac{3i}{5} f_3^{(1)} - \frac{6i}{5} f^{(0)} f_3^{(1)} - \frac{3i}{5} f^{(0)} p \frac{\partial f_3^{(1)}}{\partial p} - \frac{3i}{10} p \frac{\partial f_3^{(1)}}{\partial p} \right) \Big] \\
&+ \frac{T_e}{m_e} \left[(\mathbf{v} \cdot \mathbf{n}) \left(-\frac{7}{5} p^3 \frac{\partial^3 f^{(0)}}{\partial p^3} - \frac{47}{5} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} - \frac{15}{2} p \frac{\partial f^{(0)}}{\partial p} \right) + \right. \\
&(\hat{\mathbf{v}} \cdot \mathbf{n}) \left(\frac{6i}{5} p^2 \frac{\partial^2 f_1^{(1)}}{\partial p^2} + \frac{24i}{5} p \frac{\partial f_1^{(1)}}{\partial p} + \frac{6i}{5} f_1^{(1)} \right) + P_2 \left(-\frac{1}{2} p^2 \frac{\partial^2 f_2^{(1)}}{\partial p^2} - 2p \frac{\partial f_2^{(1)}}{\partial p} + 3f_2^{(1)} \right) \\
&\left. + P_3 \left(-\frac{3i}{10} p^2 \frac{\partial^2 f_3^{(1)}}{\partial p^2} - \frac{6i}{5} p \frac{\partial f_3^{(1)}}{\partial p} + \frac{6i}{5} f_3^{(1)} \right) + p^2 \frac{\partial^2 f_0^{(1)}}{\partial p^2} + 4p \frac{\partial f_0^{(1)}}{\partial p} \right]. \tag{2.38}
\end{aligned}$$

Vishniac effects—. The electron number density is also fluctuate. This is taken into account by imposing

$$n_e \rightarrow n_e(1 + 3\Theta_e^{(1)} + 3\Theta_e^{(2)} + 9\Theta_e^{(1)2}), \tag{2.39}$$

where Θ_e is the electron temperature perturbation. Then, we have the following additional terms:

$$\left(3\Theta_e^{(1)} + 3\Theta_e^{(2)} + 9\Theta_e^{(1)2} \right) \mathcal{C}_T^{(1)}[f], \quad 3\Theta_e^{(1)} \left(\mathcal{C}_K^{(0)}[f] + \mathcal{C}_T^{(2)}[f] \right). \tag{2.40}$$

3 Solutions to the Boltzmann equation

The cosmic microwave background radiation is almost isotropic and ideal blackbody at high precision but slightly deviates from the perfect one. In this section, we construct such a deviations as solutions of the second order Boltzmann equation.

3.1 Momentum functions

We shall start with introducing the following functional basis to expand the distribution function:

$$\mathcal{G}(p) = -p \frac{\partial f^{(0)}(p)}{\partial p}, \quad (3.1)$$

$$\mathcal{Y}(p) = \frac{1}{p^2} \frac{\partial}{\partial p} p^4 \frac{\partial f^{(0)}(p)}{\partial p}, \quad (3.2)$$

$$\mathcal{M}(p) = T_0 \frac{\partial f^{(0)}(p)}{\partial p}, \quad (3.3)$$

where we have defined

$$f^{(0)}(p) = \frac{1}{e^{\frac{p}{T_0}} - 1}. \quad (3.4)$$

The definite integrals of these functions are

$$\int_0^\infty \frac{dx}{2\pi^2} x^n f^{(0)} = \mathcal{I}_n, \quad (3.5)$$

$$\int_0^\infty \frac{dx}{2\pi^2} x^n \mathcal{G} = (n+1) \mathcal{I}_n, \quad (3.6)$$

$$\int_0^\infty \frac{dx}{2\pi^2} x^n \mathcal{Y} = (n+1)(n-2) \mathcal{I}_n, \quad (3.7)$$

$$\int_0^\infty \frac{dx}{2\pi^2} x^n \mathcal{M} = -n \mathcal{I}_{n-1}, \quad (3.8)$$

where $x = p/T_0$ and $\{I_n\}_{n=1}^3 = \{1/12, \zeta(3)/\pi^2, \pi^2/30\}$. The following relations are also useful:

$$p^2 \frac{\partial^2 f^{(0)}(p)}{\partial p^2} = \mathcal{Y} + 4\mathcal{G}, \quad (3.9)$$

$$p \frac{\partial}{\partial p} p \frac{\partial f^{(0)}(p)}{\partial p} = \mathcal{Y} + 3\mathcal{G} \quad (3.10)$$

$$-p \frac{\partial f^{(1)}(p)}{\partial p} = \Theta^{(1)}(3\mathcal{G} + \mathcal{Y}). \quad (3.11)$$

More generally,

$$p^n \frac{\partial}{\partial p} p^m \frac{\partial}{\partial p} p^l f^{(0)} = p^{n+m+l-2} \left[l(l+m-1) f^{(0)} + \mathcal{Y} + (4-m-2l) \mathcal{G} \right]. \quad (3.12)$$

These \mathcal{G} , \mathcal{Y} and \mathcal{M} are “linearly independent”. Let us consider the linear combination of these functions which is equal to zero:

$$a\mathcal{G} + b\mathcal{Y} + c\mathcal{M} = 0. \quad (3.13)$$

We then integrate the both side with respect to momentum p and obtain

$$a(n+1) \mathcal{I}_n + b(n+1)(n-2) \mathcal{I}_n - cn \mathcal{I}_{n-1} = 0. \quad (3.14)$$

The solution to (3.14) can be found to be trivial so that the \mathcal{G} , \mathcal{Y} and \mathcal{M} are linearly independent.

3.2 First order

The cosmic photon fluid perturbs along the primordial fluctuations; however, it is considered to be a blackbody at each point. Based on the assumption, we usually write the ansatz for the first order Boltzmann equation as a Planck distribution function with a spacetime dependent temperature. Let us expand the function to the linear order in terms of the temperature perturbations. Using (3.1), we can write the linear term as

$$f^{(1)} = \Theta^{(1)} \mathcal{G}, \quad (3.15)$$

where we have defined $\Theta^{(1)}$ as the first order temperature fluctuations normalized by the fiducial temperature. One then finds that the both sides of the Boltzmann equation are proportional to \mathcal{G} . This implies that the equation for $\Theta^{(1)}$ coincides with that of the energy density perturbations and the number density perturbations, which are integrated with respect to the momentum. At this stage, we succeed to justify the first assumption that the system is blackbody at each point.

3.3 Second order in the Thomson limit

On the other hand, it has been already known that the above discussions are not applicable at second order [54]. In other words, the second (or higher) order temperature perturbations are momentum dependent in general, and the fluid is no more blackbody even at each local point at second order. This is apparent if we integrate the equation with p^n . We now have three strategies to solve the Boltzmann equation. One is to integrate the momentum dependence and to write the equations for the intensity and the number density perturbations at second order. It can be significant simplification, at the same time, it masks a lot of information in the non-trivial momentum dependence of the distribution function. The second is to consider the full momentum dependent temperature. This can be perfect, but it is far more complicated. We then propose to take into account the momentum dependence partly by the form of the spectral distortions. In other words, we replace the infinite number of degrees of freedom coming from the continuous momentum with the infinite number of the parameters describing the spectral deformations. In fact, we usually apply this kind of approach to reduce the number of equations of a set of partial differential equations. For example, we use the Boltzmann hierarchy equations instead of the equations with angular parameters. In this case, the important contributions are related to the lower multipoles, and we can simplify the infinite number of equations to a few equations. Actually, we have already done this approach even for our problem at linear order. We can say that the first order spectral distortion is written as momentum independent temperature perturbations in the form of (3.15), that is, infinite number of d.o.f of continuous momentum are reduced to a single local parameter at first order. Our next step is going to second or higher order.

Let us write an arbitrary distribution function in the form of the Planck distribution function with momentum dependent temperature perturbation $\tilde{\Theta} = \tilde{\Theta}(\mathbf{x}, p\mathbf{n}, \eta)$:

$$f(\mathbf{x}, p\mathbf{n}, \eta) = \frac{1}{e^{\frac{p}{T_0} e^{-\tilde{\Theta}}} - 1}, \quad (3.16)$$

where we define T_0 as the temperature of a time independent comoving blackbody ⁴. Then let us expand this function around $\tilde{\Theta} = 0$

$$\frac{1}{e^{\frac{p}{T_0} e^{-\tilde{\Theta}}} - 1} = \sum_{n=0}^{\infty} \frac{\tilde{\Theta}^n}{n!} \left(-p \frac{\partial}{\partial p} \right)^n f^{(0)}(p). \quad (3.17)$$

In our convention, the zeroth order distribution function is time independent, and it simplifies the later calculations. Here we should notice that the n is not the order of the perturbations since the temperature perturbations have the following form:

$$\tilde{\Theta} = \sum_n \tilde{\Theta}^{(n)}, \quad (3.18)$$

where n is the order of the perturbations. We have already known that $\tilde{\Theta}^{(1)} = \Theta^{(1)}$ is momentum independent; however, $n > 1$ is momentum dependent in principle. In our definition, the higher order temperature perturbations have none zero homogeneous component since we fix the fiducial temperature T_0 as mentioned above. At second order, we find that the momentum dependence is separated as

$$\tilde{\Theta}^{(2)}(p) = \Theta^{(2)} + y \frac{\mathcal{Y}(p)}{\mathcal{G}(p)}, \quad (3.19)$$

where $\Theta^{(2)}$ is the momentum independent part and \mathcal{Y} is defined in (3.2). The perturbative expansions are obtained as follows:

$$f^{(1)} = \Theta^{(1)} \mathcal{G}(p), \quad (3.20)$$

$$f^{(2)} = \left[\Theta^{(2)} + \frac{3}{2} \Theta^2 \right] \mathcal{G}(p) + \left[y + \frac{1}{2} \Theta^2 \right] \mathcal{Y}(p), \quad (3.21)$$

where we have used (3.1) and (3.2).

3.4 Collision terms

Thanks to the expression (3.21), it simplifies calculations to write the collision terms as

$$\mathcal{C}_T^{(2)}[f] = (\mathcal{A}^{(1)} + \mathcal{A}^{(2)}) \mathcal{G} + \mathcal{B}^{(2)} \mathcal{Y}. \quad (3.22)$$

⁴ T_0 is not the average temperature which varies due to the acoustic reheating or the other heating processes at second order.

Then combining (3.22), (3.21), (2.35), (2.40) and (3.11) and we find

$$(n_e \sigma_{\text{T}} a)^{-1} \mathcal{A}^{(1)} = \mathbf{v} \cdot \mathbf{n} - \Theta + \Theta_0 - \frac{1}{2} P_2 \Theta_2, \quad (3.23)$$

$$\begin{aligned} (n_e \sigma_{\text{T}} a)^{-1} \mathcal{A}^{(2)} = & -\frac{2v^2}{5} + \mathbf{v}^{(2)} \cdot \mathbf{n} + \frac{6(\mathbf{v} \cdot \mathbf{n})^2}{5} + \mathbf{v} \cdot \mathbf{n} (\Theta + 2\Theta_0 + \Theta_2 - 2P_2 \Theta_2) \\ & + 3\Theta_e \left(\mathbf{v} \cdot \mathbf{n} - \Theta + \Theta_0 - \frac{1}{2} P_2 \Theta_2 \right) - iv \left[-\Theta_1 + \frac{1}{5} P_2 \left(-4\Theta_1 - \frac{3}{2} \Theta_3 \right) \right] \\ & + \Theta_0^{(2)} - \Theta^{(2)} + \frac{3}{2} [\Theta^2]_0 - \frac{3}{2} \Theta^2 + \frac{1}{10} \sum_{m=-2}^2 \left(\frac{3}{2} [\Theta^2]_{2m} + \Theta_{2m}^{(2)} \right) Y_{2m}, \end{aligned} \quad (3.24)$$

$$\begin{aligned} (n_e \sigma_{\text{T}} a)^{-1} \mathcal{B}^{(2)} = & \frac{3}{20} v^2 + \frac{11}{20} (\mathbf{v} \cdot \mathbf{n})^2 - \frac{\Theta^2}{2} + \frac{[\Theta^2]_0}{2} + \mathbf{v} \cdot \mathbf{n} \left(\Theta_0 - \frac{1}{2} P_2 \Theta_2 \right) \\ & - iv \left[-\Theta_1 + \frac{1}{5} P_2 \left(-\Theta_1 + \frac{3}{2} \Theta_3 \right) \right] \\ & + \frac{1}{20} \sum_{m=-2}^2 [\Theta^2]_{2m} Y_{2m} \\ & - y + y_0 + \frac{1}{10} \sum_{m=-2}^2 y_{2m} Y_{2m}. \end{aligned} \quad (3.25)$$

Note that the pure second order quantities such as $\Theta^{(2)}$ and $\mathbf{v}^{(2)}$ only appear in $\mathcal{A}^{(2)}$, and $\mathcal{B}^{(2)}$ is expressed by products of first order perturbations except y . The monopole component of $\mathcal{A}^{(2)}$ is calculated as zero. This implies that the Compton scattering does not change the isotropic component of the photon number density even at second order.

3.5 Liouville terms

We will solve the above equation perturbatively; however, note that we avoid writing the second order metric perturbations explicitly in the following discussions since they do not appear in the final expressions for the spectral distortions. So far we have discussed the r.h.s. of the Boltzmann equation. Next, let us see the Liouville term on the left. Differentiating the distribution function with respect to the conformal time, (3.21) yields

$$\begin{aligned} f' = & f^{(0)'} + (\Theta' + 3\Theta\Theta') \mathcal{G} + \left(\Theta + \frac{3}{2} \Theta^2 \right) \mathcal{G}' \\ & + (y' + \Theta\Theta') \mathcal{Y} + \left(y + \frac{1}{2} \Theta^2 \right) \mathcal{Y}', \end{aligned} \quad (3.26)$$

where $' \equiv d/d\eta$. Since $f^{(0)}$ depends only on comoving momentum p , the derivative of the zeroth order part becomes

$$f^{(0)'} = -(\ln p)' \mathcal{G}. \quad (3.27)$$

Note that $-(\ln p)'$ does not have the zeroth order part but both the first and the second order terms which describe the gravitational redshift and lensing since p is the comoving momentum. On the other hand, using (3.10) we can write the time derivative of \mathcal{G} by

$$\mathcal{G}' = -(\ln p)'(3\mathcal{G} + \mathcal{Y}), \quad (3.28)$$

and \mathcal{Y}' can be neglected since this is the first order quantity which is multiplied with the second order perturbations in the equation. Combining (3.26), (3.27) and (3.28) up to second order, we obtain

$$f' = (1 + 3\Theta)[\Theta' - (\ln p)']\mathcal{G} + (y' + \Theta[\Theta' - (\ln p)'])\mathcal{Y}. \quad (3.29)$$

3.6 Second order equations for the spectral distortion and acoustic reheating

Next, let us write down equations order by order. The first order Boltzmann equation is easily obtained as

$$\Theta^{(1)'} - (\ln p)^{(1)'} = \mathcal{A}^{(1)}. \quad (3.30)$$

If we expand $(\ln p)^{(1)'}$ with respect to the metric perturbations, we obtain first order Boltzmann equation of the temperature perturbations with metric perturbations. On the other hand, collecting second order terms, the second order equation can be written as

$$\left[\Theta^{(2)'} - (\ln p)^{(2)'} + 3\Theta\mathcal{A}^{(1)} \right] \mathcal{G} + \left[y' + \Theta\mathcal{A}^{(1)} \right] \mathcal{Y} = \mathcal{A}^{(2)}\mathcal{G} + \mathcal{B}^{(2)}\mathcal{Y}, \quad (3.31)$$

where we have used (3.30) for substituting $\mathcal{A}^{(1)}$ into the expression. As we commented above, $\mathcal{A}^{(2)}$ does not have any monopole terms since the Compton scattering does not change the photon number; however, the temperature is raised by acoustic reheating, namely, the photon number is changed due to $3\Theta\mathcal{A}^{(1)}$ on the left. Integrals with p^n should be always consistent even if they do not have any physical implications since we should respect the equation at the distribution function level. Therefore, each coefficient for \mathcal{G} and \mathcal{Y} should be equal so that we obtain the Boltzmann equations for $\Theta^{(2)}$ and y independently. One then immediately finds

$$\left[\Theta^{(2)'} - (\ln p)^{(2)'} + 3\Theta\mathcal{A}^{(1)} \right] = \mathcal{A}^{(2)}, \quad (3.32)$$

$$y' + \Theta\mathcal{A}^{(1)} = \mathcal{B}^{(2)}. \quad (3.33)$$

In contrast to (3.30), there are source terms in the l.h.s. of (3.32), and this implies that the small scale perturbation generates large scale temperature perturbations at second order, whose homogenous component is recently pointed out in [58, 59]. It is important that $\mathcal{B}^{(2)}$ does not have any pure second order terms except y . If one does not introduce the y to the distribution function at the beginning, (3.31) is not satisfied since both $\Theta\mathcal{A}^{(1)}$ and

$\mathcal{B}^{(2)}$ are already determined at first order and do not coincide in principle. This also implies that the y is determined by the linear perturbations automatically. Let us expand (3.33) by substituting the following form:

$$\mathcal{A}^{(1)} = -n_e \sigma_T a \left(\Theta - \Theta_0 - \mathbf{v} \cdot \mathbf{n} + \frac{1}{2} P_2 \Theta_2 \right), \quad (3.34)$$

$$\begin{aligned} \mathcal{B}^{(2)} = & -n_e \sigma_T a \left[\frac{\Theta^2}{2} - \frac{[\Theta^2]_0}{2} - \frac{3}{20} v^2 - \frac{11}{20} (\mathbf{v} \cdot \mathbf{n})^2 \right. \\ & - \frac{1}{20} \sum_{m=-2}^2 [\Theta^2]_{2m} Y_{2m} - \mathbf{v} \cdot \mathbf{n} \left(\Theta_0 - \frac{1}{2} P_2 \Theta_2 \right) \\ & + i v \left\{ -\Theta_1 + \frac{1}{5} P_2 \left(-\Theta_1 + \frac{3}{2} \Theta_3 \right) \right\} \\ & \left. + y - y_0 - \frac{1}{10} \sum_{m=-2}^2 y_{2m} Y_{2m} \right] \end{aligned} \quad (3.35)$$

Then we find

$$\begin{aligned} \frac{\partial y}{\partial \eta} + \mathbf{n} \cdot \nabla y = & n_e \sigma_T a \left(\Theta - \Theta_0 - \mathbf{v} \cdot \mathbf{n} + \frac{1}{2} P_2 \Theta_2 \right) \Theta^{(1)} \\ & - n_e \sigma_T a \left[\frac{\Theta^2}{2} - \frac{[\Theta^2]_0}{2} - \frac{3}{20} v^2 - \frac{11}{20} (\mathbf{v} \cdot \mathbf{n})^2 \right. \\ & - \frac{1}{20} \sum_{m=-2}^2 [\Theta^2]_{2m} Y_{2m} - \mathbf{v} \cdot \mathbf{n} \left(\Theta_0 - \frac{1}{2} P_2 \Theta_2 \right) \\ & \left. + i v \left\{ -\Theta_1 + \frac{1}{5} P_2 \left(-\Theta_1 + \frac{3}{2} \Theta_3 \right) \right\} \right] \\ & - n_e \sigma_T a \left(y - y_0 - \frac{1}{10} \sum_{m=-2}^2 y_{2m} Y_{2m} \right). \end{aligned} \quad (3.36)$$

On the other hand, the Fourier transform of the above equation becomes

$$\begin{aligned} \frac{\partial y}{\partial \eta} + i k \lambda y = & n_e \sigma_T a \left(\Theta - \Theta_0 - v \lambda + \frac{1}{2} P_2 \Theta_2 \right) \Theta^{(1)} \\ & - n_e \sigma_T a \left[\frac{\Theta^2}{2} - \frac{[\Theta^2]_0}{2} - \frac{3}{20} v^2 - \frac{11}{20} (\lambda v)^2 \right. \\ & - \frac{1}{20} \sum_{m=-2}^2 [\Theta^2]_{2m} Y_{2m} - \lambda v \left(\Theta_0 - \frac{1}{2} P_2 \Theta_2 \right) \\ & \left. + i v \left\{ -\Theta_1 + \frac{1}{5} P_2 \left(-\Theta_1 + \frac{3}{2} \Theta_3 \right) \right\} \right] \\ & - n_e \sigma_T a \left(y - y_0 - \frac{1}{10} \sum_{m=-2}^2 y_{2m} Y_{2m} \right), \end{aligned} \quad (3.37)$$

where λ is the cosine between the Fourier momentum and the photon momentum, and products of perturbations are understood as the convolutions. Here we also write the $m \neq 0$ components. The equations for the spectral distortions do not include the other pure second order quantities such as the temperature and the metric perturbations. Therefore, we do not have to integrate the full second order Boltzmann equation as long as working only on the spectral distortions. Note that the convolutions include the curvature perturbations implicitly as discussed in the appendix C. Therefore, the integration with respect to the Fourier momentum is non-trivial in general.

4 Inhomogeneous y distortion

We have introduced y distortion as momentum dependent part of the second order temperature perturbations. Its momentum dependence is the same form with the usual Compton y parameter given as

$$y_C = \int \frac{T_e - T_\gamma}{m_e} n_e \sigma_T a d\eta, \quad (4.1)$$

where $T_\gamma \equiv T_0(1+z)$. We should note that our y is a free parameter determined by the second order Boltzmann equation and has nothing to do with the inhomogeneity of T_e , T_γ and n_e in the integrand in (4.1). To be more specific, our y arises from the expansion associated with the Thomson collision terms but y_C comes from the Kompaneets terms, namely, their momentum dependences coincide accidentally. Below we summarize the evolution equation for our y and confirm the availability of the previous estimations.

4.1 Hierarchy equations for the spectral distortion

In this section, we write the hierarchy equations for multipole components of the distortion. Here we ignore $m \neq 0$ (vector and tensor) components for simplicity. (3.37) is then written as

$$\dot{y} + ik\lambda y = \mathcal{S}(k, \lambda) - n_e \sigma_T a \left(y - y_0 + \frac{1}{2} P_2(\lambda) y_2 \right), \quad (4.2)$$

where $\dot{\cdot} \equiv \partial/\partial\eta$ and the source term is defined by

$$\begin{aligned} (n_e \sigma_T a)^{-1} \mathcal{S}(k, \lambda) = & \left(\Theta - \Theta_0 - v\lambda + \frac{1}{2} P_2 \Theta_2 \right) \Theta^{(1)} \\ & - \frac{\Theta^2}{2} + \frac{[\Theta^2]_0}{2} + \frac{3}{20} v^2 + \frac{11}{20} (\lambda v)^2 \\ & - \frac{1}{4} [\Theta^2]_2 P_2 + \lambda v \left(\Theta_0 - \frac{1}{2} P_2 \Theta_2 \right) \\ & - iv \left\{ -\Theta_1 + \frac{1}{5} P_2 \left(-\Theta_1 + \frac{3}{2} \Theta_3 \right) \right\} \end{aligned} \quad (4.3)$$

Using (A.1) we obtain the following hierarchy equations for the y distortions:

$$\dot{y}_l + \frac{k(l+1)}{2l+1}y_{l+1} - \frac{kl}{2l+1}y_{l-1} = \mathcal{S}_l - n_e\sigma_{\text{T}}a \left(1 - \delta_{l0} - \frac{1}{10}\delta_{2l}\right)y_l. \quad (4.4)$$

Up to $l = 4$, the source functions can be expanded as

$$(n_e\sigma_{\text{T}}a)^{-1}\mathcal{S}_0 = \frac{v^2}{3} + 2iv\Theta_1 - 3\Theta_1^2 + \frac{9\Theta_2^2}{2} - 7\Theta_3^2 + 9\Theta_4^2 + \dots \quad (4.5)$$

$$(n_e\sigma_{\text{T}}a)^{-1}\mathcal{S}_1 = \frac{3}{5}i\Theta_2v - \frac{9}{5}\Theta_1\Theta_2 + \frac{27}{10}\Theta_2\Theta_3 - 4\Theta_3\Theta_4 \dots \quad (4.6)$$

$$\begin{aligned} (n_e\sigma_{\text{T}}a)^{-1}\mathcal{S}_2 = & -\frac{11}{150}v^2 - \frac{11}{25}i\Theta_1v + \frac{33}{50}i\Theta_3v + \frac{33}{50}\Theta_1^2 - \frac{9}{14}\Theta_2^2 \\ & - \frac{99}{50}\Theta_1\Theta_3 + \frac{77}{75}\Theta_3^2 + \frac{18}{7}\Theta_2\Theta_4 - \frac{9}{7}\Theta_4^2 \dots \end{aligned} \quad (4.7)$$

Let k be the super horizon scales. The $l > 0$ linear perturbations are not significant before the horizon entry. Therefore, using (C.5) each convolution should be well approximated as

$$(XY)_{\mathbf{k}} \sim \int \frac{dq}{q} X_q Y_q \mathcal{P}_{\mathcal{R}}(q). \quad (4.8)$$

(4.8) implies that the source terms induce k independent transfer functions for the y distortion on large scales. Imposing $v = -3i\Theta_1$ during tight coupling regime, we obtain

$$(n_e\sigma_{\text{T}}a)^{-1}\mathcal{S}_0 = \frac{9}{2}\Theta_2^2 - 7\Theta_3^2 + 9\Theta_4^2 + \dots \quad (4.9)$$

$$(n_e\sigma_{\text{T}}a)^{-1}\mathcal{S}_1 = \frac{27}{10}\Theta_3\Theta_2 - 4\Theta_3\Theta_4 + \dots \quad (4.10)$$

$$(n_e\sigma_{\text{T}}a)^{-1}\mathcal{S}_2 = -\frac{9}{14}\Theta_2^2 + \frac{18}{7}\Theta_4\Theta_2 + \frac{77}{75}\Theta_3^2 - \frac{9}{7}\Theta_4^2 + \dots \quad (4.11)$$

Ignoring the gradient terms, we have

$$\dot{y}_0 \sim n_e\sigma_{\text{T}}a \left(\frac{9}{2}\Theta_2^2 - 7\Theta_3^2 + 9\Theta_4^2 + \dots \right) \quad (4.12)$$

$$\dot{y}_1 \sim -n_e\sigma_{\text{T}}ay_1 + n_e\sigma_{\text{T}}a \left(\frac{27}{10}\Theta_3\Theta_2 - 4\Theta_3\Theta_4 + \dots \right) \quad (4.13)$$

$$\dot{y}_2 \sim -n_e\sigma_{\text{T}}ay_2 + n_e\sigma_{\text{T}}a \left(-\frac{9}{14}\Theta_2^2 + \frac{18}{7}\Theta_4\Theta_2 + \frac{77}{75}\Theta_3^2 - \frac{9}{7}\Theta_4^2 + \dots \right). \quad (4.14)$$

The first term in r.h.s. of (4.12) is well known. Θ_2 implies the emergence of the anisotropic stress which induces the friction heat, and it sources the distortion. $-n_e\sigma_{\text{T}}ay_2$ suppresses growing of y_2 by isotropization due to the Thomson scattering, and this is the same for the higher order multipoles. The term proportional to Θ_2^2 in (4.14) implies that the distortions also diffuse due to the anisotropic stress. This effect was previously taken into account by the window function introduced by hands.

Substituting $l = 0$ into (4.4) and ignoring the gradient terms, we immediately find that only the monopole component of the y distortions can survive and is conserved at super horizon after y era. The hierarchy equations at late periods without the sources are given by

$$\dot{y}_l + \frac{k(l+1)}{2l+1}y_{l+1} - \frac{kl}{2l+1}y_{l-1} = -n_e\sigma_T a \left(1 - \delta_{l0} - \frac{1}{10}\delta_{2l}\right)y_l, \quad (4.15)$$

and from (4.12), the initial condition of the distortion should be written as

$$\begin{aligned} y_0(\eta_f, k) &\sim \int_{\eta_i}^{\eta_f} n_e\sigma_T a \left(\frac{v^2}{3} + 2iv\Theta_1 - 3\Theta_1^2 + \frac{9\Theta_2^2}{2} - 7\Theta_3^2 + 9\Theta_4^2 + \dots \right) d\eta \\ &\sim \int_{\eta_i}^{\eta_f} n_e\sigma_T a \left(\frac{9}{2}\Theta_2^2 - 7\Theta_3^2 + 9\Theta_4^2 + \dots \right) d\eta. \end{aligned} \quad (4.16)$$

Thanks to (4.8), the main part of $y_0(\eta_f, k)$ is completely the same form with the homogeneous component previously evaluated, for example, in [37]. This is reasonable since the monopole and homogeneous part are not distinguishable before horizon entry.

4.2 Integral solutions and Gauge dependence

We shall now demonstrate a line-of-sight integral method for the y distortion. Near the last scattering surface, the source can be negligible and the equations can be written as

$$\dot{y} + ik\lambda y = -n_e\sigma_T a \left(y - y_0 + \frac{1}{2}P_2(\lambda)y_2 \right). \quad (4.17)$$

The method is completely parallel with that for the temperature perturbations. That is, the line-of-sight integral solution for the y distortion is given by

$$y(k, \lambda, \eta_0) = \int_{\eta_f}^{\eta_0} d\eta \mathcal{S}_y(k, \eta) e^{-ik(\eta_0 - \eta)\lambda} \quad (4.18)$$

$$\mathcal{S}_y(k, \eta) = g \left(y_0 + \frac{y_2}{4} \right) + \ddot{g} \frac{3y_2}{4k^2}, \quad (4.19)$$

where the visibility function is introduced by $g = -\dot{\tau}e^{-\tau}$ with $\dot{\tau}$ being $-n_e\sigma_T a$. The terms related to y_2 are new corrections. The harmonic coefficient is also immediately obtained as

$$a_{y,lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} Y_{lm}^*(\hat{k}) \int_0^{\eta_0} d\eta \mathcal{S}_y(k, \eta) j_l[k(\eta_0 - \eta)]. \quad (4.20)$$

We can see that no metric perturbations and no second order temperature perturbations are included. Therefore, y has no redshift and no crosscorrelation with the ISW lensing. This helps us to consider the $\mu\mu$ and yy auto and μy cross correlations since we do not have to consider the curve of sight [23].

In the end of this section, we shall comment on the gauge dependence of y . The gauge transformation laws for v and $-3i\Theta_1$ are the same as given in [60], and higher order multipoles are gauge invariant quantities. Therefore, gauge invariance of y is manifest since the velocity terms in the first line of (4.16) can be recast into $(v + 3i\Theta_1)^2/3$ [37]. There is no metric perturbation in (3.37) so that y distortion evolves gauge independently after its generation.

4.3 Homogeneous component of the y

In our definition, y is calculated independently from the usual Compton y parameter defined in (4.1). Let us combine these two y distortions. By using Compton y parameter, (2.36) is written as

$$C_K^{(0)}[f] = \dot{y}_C \mathcal{Y}(p). \quad (4.21)$$

The ensemble average of the monopole component of the y becomes

$$\frac{\partial \langle y_{\text{tot}} \rangle}{\partial \eta} = n_e \sigma_T a \left[\frac{\langle v^2 \rangle}{3} + 2i \langle v \Theta_1 \rangle - 3 \langle \Theta_1^2 \rangle + \frac{9 \langle \Theta_2^2 \rangle}{2} - 7 \langle \Theta_3^2 \rangle + 9 \langle \Theta_4^2 \rangle + \dots \right] + \dot{y}_C. \quad (4.22)$$

Therefore the total homogeneous component can be calculated as

$$\langle y_{\text{tot}} \rangle = - \int \dot{\tau} \left[\frac{T_e - T_\gamma}{m_e} + \frac{\langle v^2 \rangle}{3} + 2i \langle v \Theta_1 \rangle - 3 \langle \Theta_1^2 \rangle + \frac{9 \langle \Theta_2^2 \rangle}{2} - 7 \langle \Theta_3^2 \rangle + 9 \langle \Theta_4^2 \rangle + \dots \right] d\eta, \quad (4.23)$$

where the baryon bulk velocity and the temperature dipole are cancel if we apply the tight coupling approximation, namely $v = -3i\Theta_1$. On the other hand, SZ effect can be also calculated in the above formula if we impose $T_e \gg T_\gamma$ and $v \gg \Theta$.

5 μ distortion

5.1 Definition

We have shown that the y is necessary for a set of equations to be consistent at second order; however we have not comment on the chemical potential type distortion called μ distortion. During $5 \times 10^4 < z < 2 \times 10^6$, the y distortions are converted to the μ distortion, and the system is considered to be in kinetic equilibrium. This was investigated numerically in the previous studies [34, 37]. Let us try to include the μ as well in our formulation.

One strategy to include the chemical potential may be writing a second order ansatz in the following form:

$$\tilde{\Theta}^{(2)}(p) = \Theta^{(2)} + y \frac{\mathcal{Y}(p)}{\mathcal{G}(p)} + \mu \frac{\mathcal{M}(p)}{\mathcal{G}(p)} + \dots, \quad (5.1)$$

where \mathcal{M} is defined in (3.3). Let us substitute (5.1) into (2.35) and (3.29). We then obtain additional terms proportional to \mathcal{M} . Reading off each coefficient, the evolution equation for the μ distortion is given as follows:

$$\dot{\mu} + ik\lambda\mu = (n_e \sigma_T a) \left[-\mu + \mu_0 + \frac{1}{10} \sum_{m=-2}^2 \mu_{2m} Y_{2m} \right]. \quad (5.2)$$

It is not surprising that the conversion of y to μ is not seen even at second order since we start with a momentum independent μ parameter, and we did not take into account the

momentum transfer. (5.2) just tells us that the momentum independent chemical potential evolves independently from the y distortion and the second order temperature perturbations once it is given at initial time. This implies that μ generation from y should be treated in the full (or higher order) Boltzmann equations with momentum dependent chemical potential. The other steps for the μ are completely parallel with those for the y , and the harmonic coefficient is the same form with (4.20), namely we have

$$a_{\mu,lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} Y_{lm}^*(\hat{k}) \int_0^{\eta_0} d\eta \mathcal{S}_\mu(k, \eta) j_l[k(\eta_0 - \eta)], \quad (5.3)$$

$$\mathcal{S}_\mu(k, \eta) = g \left(\mu_0 + \frac{\mu_2}{4} \right) + \ddot{g} \frac{3\mu_2}{4k^2}, \quad (5.4)$$

and the initial value of the μ distortion should be introduced by hands in this context.

5.2 Instantaneous μ formation

The full numerical analysis for μ generation is complicated. Here we notice that the chemical potential is a thermodynamic quantity and that we do not have to care about the details of the process when considering the thermalization time scale is rapid enough. In this section we shall repeat a traditional explanation for the μ formation with a single comment. Let us consider that the initial state is given by the solutions of the Thomson limit second order Boltzmann equation, namely, the second order number and the energy density are calculated as

$$N_y^{(2)} = 0, \quad (5.5)$$

$$I_y^{(2)} = 4y\mathcal{I}_3, \quad (5.6)$$

where numerical factors \mathcal{I}_n are defined in (3.5). Assuming that the thermalization time scale is rapid enough compared to the typical that of the cosmic expansion, these quantities should have the following forms at the next moment:

$$N_{\text{BE}}^{(2)} = 3\mathcal{I}_2\Theta_{\text{BE}}^{(2)} - 2\mu\mathcal{I}_1 \quad (5.7)$$

$$I_{\text{BE}}^{(2)} = 4\mathcal{I}_3\Theta_{\text{BE}}^{(2)} - 3\mu\mathcal{I}_2 \quad (5.8)$$

where subscript “BE” implies that they are the parameters associated with a Bose distribution function. Then we can impose the number and the energy conservation laws:

$$N_y^{(2)} = N_{\text{BE}}^{(2)} \quad (5.9)$$

$$I_y^{(2)} = I_{\text{BE}}^{(2)} \quad (5.10)$$

so that we obtain

$$\mu = \left(\frac{2\mathcal{I}_1}{3\mathcal{I}_2} - \frac{3\mathcal{I}_2}{4\mathcal{I}_3} \right)^{-1} y. \quad (5.11)$$

The numerical constant is calculated as $\mu = 1.40066 \times 4y$, and the well known relation is derived. Now we shall have a comment on this matter. (5.9) and (5.10) should not be established at a distribution function level, namely, the continuous evolution of the μ is never explained by this approach. This is because the momentum integrals with p^n should be always consistent if we start with the Boltzmann equation. Suppose that, for instance, we use the Boltzmann equations which are integrated with p^4 and p^5 , we find the other numerical factor in (5.11). Therefore, there exist time discontinuities in the both sides of the equalities in (5.9) and (5.10). Using (4.16) and (5.11) we approximately obtain the following form of μ distortion from the scalar perturbations:

$$\mu_0(\eta_f, k) \sim \int_{\eta_i}^{\eta_f} n_e \sigma_T a \left(\frac{9}{2} \Theta_2^2 - 7 \Theta_3^2 + 9 \Theta_4^2 + \dots \right) d\eta. \quad (5.12)$$

5.3 Monopole formation

We have mentioned that the μ formation is non-trivial and our prescription is not applicable. Here let us revisit the monopole terms in (2.38). These terms are actually coincide with those in (2.36). This implies that the previous numerical simulation based on (2.36) is also applicable to the inhomogeneous case as long as we ignore the higher order multipoles. As discussed in the previous section, multipoles of the spectral distortions are vanishing, and only the long wavelength modes of the monopole component are dominant. In this sense, we expect that the generation of the inhomogeneous distortions should be explained in the same manner with the previous numerical calculations for the homogenous μ distortions [34, 37].

5.4 Suppression from the Double Compton scattering

So far we have introduced the initial redshift for the μ by referring to the previous numerical works [34, 37]. It is determined by the double Compton effect, which is cubic order QED interaction. The process is crucial for the spectral distortions since it changes the number of photons and erases the distortions. The derivation of the double Compton scattering collision term is complicated so that we avoid writing the term explicitly here. Instead, roughly we estimate the time scales of these interactions. Let Γ_K and Γ_{DC} be the energy transfer ratio due to the Compton scattering and double Compton scattering, respectively. Γ_K should be proportional to the scattering event ratio $n_e \sigma_T$. Note that there is no energy transfer only if $n_e \sigma_T$ is large. There exist significant energy transfers if electrons are relativistic enough to transfer the photon energy. This should be characterized by T_e/m_e . Then, we can estimate the energy transfer ratio due to the Compton scattering as follows:

$$\Gamma_K \sim n_e \sigma_T \frac{T_\gamma}{m_e}. \quad (5.13)$$

Actually, we can reproduce this relation by using (2.36). In analogy with the above discussion, we can roughly estimate Γ_{DC} as well. First, the double Compton scattering interaction ratio

should be proportional to $n_e \sigma_T \alpha \epsilon^2$ with α being the fine structure constant. This is because the process is a cubic order QED interaction, and $\epsilon \rightarrow 0$ limit electron does not emit the second photon in terms of energy conservation law⁵. Then, the lowest order term can be estimated as

$$\Gamma_{\text{DC}} \sim n_e \sigma_T \alpha \left(\frac{T_\gamma}{m_e} \right)^2. \quad (5.14)$$

Using Γ_K and Γ_{DC} , we can guess the suppression time scale of the μ distortion. The μ distortion varies as a result of both the double Compton scattering and the Compton scattering. Therefore, the suppression time scale can be given as the inverse of $\Gamma_\mu \sim \sqrt{\Gamma_{\text{DC}} \Gamma_K}$. Employing these facts, we roughly obtain $\Gamma_\mu \sim 10^{-35} \times (1+z)^{\frac{9}{2}} \text{s}^{-1}$ and $\Gamma_K \sim 10^{-29} \times (1+z)^4 \text{s}^{-1}$. Comparing these with $H \sim 10^{-20} \times (1+z)^2 \text{s}^{-1}$, one finds that the window of μ era is opened during $\mathcal{O}(10^5) < z < \mathcal{O}(10^6)$.

6 Higher order spectral distortions

A product of distribution functions in (2.18) can be expanded as

$$\begin{aligned} & g(\tilde{\mathbf{q}}') f(\tilde{\mathbf{p}}') [1 + f(\tilde{\mathbf{p}})] - g(\tilde{\mathbf{q}}) f(\tilde{\mathbf{p}}) [1 + f(\tilde{\mathbf{p}}')] \\ &= g(\tilde{\mathbf{q}}) \left(f(\tilde{\mathbf{p}}') - f(\tilde{\mathbf{p}}) + \left(-\frac{(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')^2}{2m_e T_e} - \frac{(\tilde{\mathbf{q}} - m_e \mathbf{v}) \cdot (\tilde{\mathbf{p}} - \tilde{\mathbf{p}}')}{m_e T_e} \right) f(\tilde{\mathbf{p}}') [1 + f(\tilde{\mathbf{p}})] + \dots \right) \\ &= g(\tilde{\mathbf{q}}) \left[f(\tilde{\mathbf{p}}') - f(\tilde{\mathbf{p}}) + \mathcal{O}\left(\frac{\eta}{\epsilon}\right) \right]. \end{aligned} \quad (6.1)$$

This implies that the collision terms are linear in f if we ignore the momentum transfer corrections coming from \tilde{p}/m_e and T_e/m_e . We start this section with the above Thomson scattering limit.

6.1 Cubic order ansatz at Thomson limit

The dimensional quantity is only the photon momentum in the collision terms. Therefore, the derivative operators always appear in the form of $p \partial / \partial p$. The cubic order Thomson term $(n_e \sigma_T a)^{-1} \mathcal{C}_T^{(3)}[f]$ should be written as a linear combination of

$$f, \quad p \frac{\partial f}{\partial p}, \quad \left(p \frac{\partial}{\partial p} \right)^2 f, \quad \left(p \frac{\partial}{\partial p} \right)^3 f, \quad (6.2)$$

and their Legendre coefficients with the baryon bulk velocity as explicitly shown in (2.37). Let us introduce a following momentum function:

$$\mathcal{K}(p) = \left(-p \frac{\partial}{\partial p} \right) \mathcal{Y}(p), \quad (6.3)$$

⁵ Linear terms in ϵ do not exist since the scattering cross section should be a Lorentz scalar.

where the momentum integral of \mathcal{K} with p^2 is 0, which inspires us to define a higher order y distortion. Using this function, the third order derivative of the Planck distribution is given as

$$\left(-p \frac{\partial}{\partial p}\right)^3 f^{(0)}(p) = \mathcal{K} + 3\mathcal{Y} + 9\mathcal{G}. \quad (6.4)$$

Combining the above with (3.17), one finds the following cubic order terms:

$$f^{(3)} = \tilde{\Theta}^{(3)}\mathcal{G} + \tilde{\Theta}^{(1)}\tilde{\Theta}^{(2)}(3\mathcal{G} + \mathcal{Y}) + \frac{\tilde{\Theta}^{(1)3}}{3!}(9\mathcal{G} + 3\mathcal{Y} + \mathcal{K}). \quad (6.5)$$

Then we separate the momentum dependence of the temperature perturbations as

$$\tilde{\Theta}^{(1)}(p) = \Theta^{(1)} \quad (6.6)$$

$$\tilde{\Theta}^{(2)}(p) = \Theta^{(2)} + \frac{\mathcal{Y}}{\mathcal{G}}y^{(2)} \quad (6.7)$$

$$\tilde{\Theta}^{(3)}(p) = \Theta^{(3)} - \frac{\mathcal{Y}^2}{\mathcal{G}^2}\Theta^{(1)}y^{(2)} + \frac{\mathcal{Y}}{\mathcal{G}}y^{(3)} + \frac{\mathcal{K}}{\mathcal{G}}\kappa^{(3)}, \quad (6.8)$$

and we can write the cubic order terms as

$$\begin{aligned} f^{(3)} = & \left[\Theta^{(3)} + 3\Theta^{(1)}\Theta^{(2)} + \frac{3}{2}\Theta^{(1)3} \right] \mathcal{G} \\ & + \left[\Theta^{(1)}\Theta^{(2)} + \frac{1}{2}\Theta^{(1)3} + 3\Theta^{(1)}y^{(2)} + y^{(3)} \right] \mathcal{Y} \\ & + \left[\frac{1}{3!}\Theta^{(1)3} + \kappa^{(3)} \right] \mathcal{K}. \end{aligned} \quad (6.9)$$

This is our ansatz for the cubic order Thomson limit Boltzmann equation. The former discussion suggests that closed equations for the higher order spectral distortions such as $\Theta^{(3)}$, $y^{(3)}$ and $\kappa^{(3)}$ are systematically obtained. We can reconstruct the distribution functions in the form of the sum of local blackbody and spectral distortions. (6.5) and (6.9) yield

$$f = \frac{1}{e^{\frac{p}{T_0}} e^{-\Theta} - 1} + \left[(1 + 3\Theta^{(1)})y^{(2)} + y^{(3)} \right] \mathcal{Y} + \kappa^{(3)}\mathcal{K}, \quad (6.10)$$

where we have defined momentum independent temperature perturbation as $\Theta = \Theta^{(1)} + \Theta^{(2)} + \Theta^{(3)}$. The spectral shapes of the momentum basis are shown in Fig.1. Defining $\alpha = 2\mathcal{I}_1/(3\mathcal{I}_2)$, conventionally the μ distortion is expressed by not \mathcal{M} but $\mathcal{M} + \alpha\mathcal{G}$, which is the difference between a Bose and a Planck distributions whose number densities are the same. $y^{(3)}$ can be subdominant part of $y^{(2)}$; however, we can identify $\kappa^{(3)}$ due to the momentum dependence even if its magnitude is smaller.

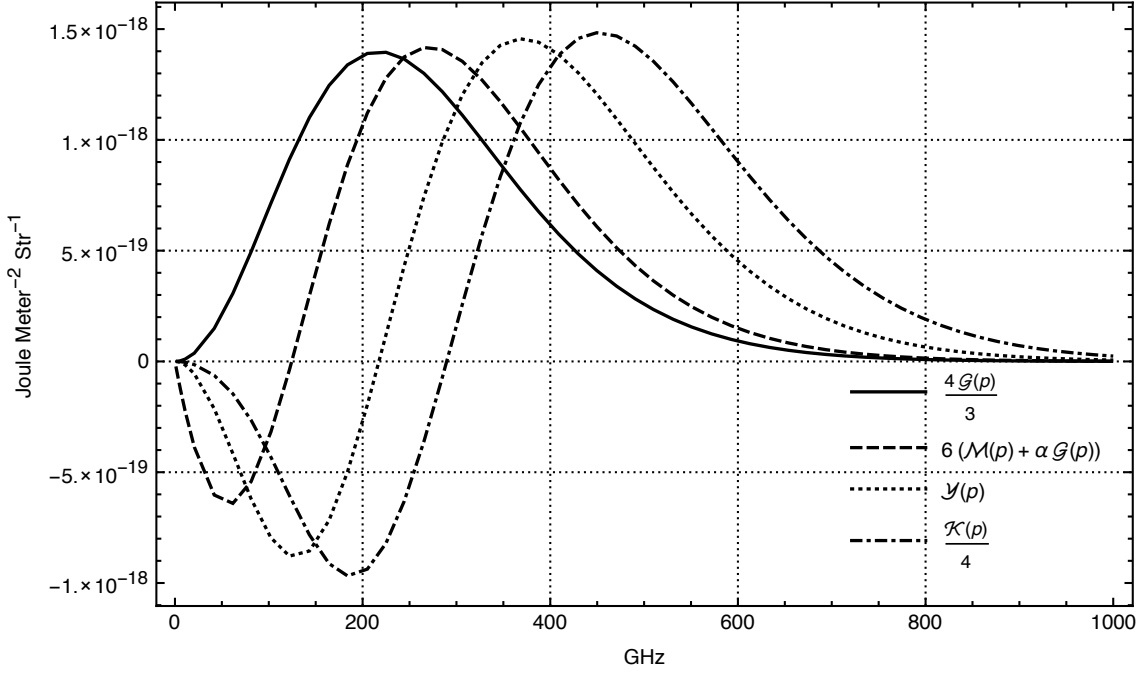


Figure 1. Spectral shapes of the photon number shift, μ distortion, y distortion and higher order y distortion are drawn. They are rescaled for comparing the shapes and the peaks. The multiples are shown in the legend in the figure.

6.2 General ansatz at Thomson limit

The same prescription is available for higher orders as long as we assume the linearity of the distribution functions. Let us introduce n -th order momentum function whose integral with p^2 is zero:

$$\mathcal{Y}^{(n+1)}(p) = \left(-p \frac{\partial}{\partial p}\right)^n \mathcal{Y}(p), \quad (6.11)$$

where $\mathcal{Y}^{(1)} = \mathcal{Y}$ and $\mathcal{Y}^{(2)} = \mathcal{K}$. Then, l -th term in (3.17) is expressed as

$$\left(-p \frac{\partial}{\partial p}\right)^l f^{(0)}(p) = \mathcal{Y}^{(l-1)} + 3\mathcal{Y}^{(l-2)} + \dots + 3^{l-2}\mathcal{Y}^{(1)} + 3^{l-1}\mathcal{G} \quad (6.12)$$

$$= 3^{l-1}\mathcal{G}(p) + \sum_{k=1}^{l-1} 3^{l-k-1}\mathcal{Y}^{(k)}(p). \quad (6.13)$$

As discussed in (6.2), the momentum dependence is always expressed by linear combination of \mathcal{G} , $\mathcal{Y}^{(1)}$, \dots and $\mathcal{Y}^{(n-1)}$. Using these functions, the n -th order distribution function should be written as

$$f^{(n)} = \left[\Theta^{(n)} + \dots\right] \mathcal{G} + \dots + \left[\dots + y^{(n-1,1)}\right] \mathcal{Y}^{(n-2)} + \left[\frac{1}{n!} \left(\Theta^{(1)}\right)^n + y^{(n,0)}\right] \mathcal{Y}^{(n-1)}, \quad (6.14)$$

where $y^{(2)} = y^{(2,0)}$, $y^{(3)} = y^{(2,1)}$ and $\kappa^{(3)} = y^{(3,0)}$. A number of the new parameters for the n -th order Thomson limit Boltzmann equations can be n . On the other hand, the time derivative of the momentum basis is calculated as

$$\mathcal{Y}^{(n)'} = -(\ln p)' \mathcal{Y}^{(n+1)}. \quad (6.15)$$

Using this with the same manipulation for (3.29), acoustic sources for higher order distortions can be written as

$$y^{(n,0)'} = -\frac{1}{(n-1)!} \Theta^{(1)n-1} \mathcal{A}^{(1)} + \dots. \quad (6.16)$$

Therefore, we always have the higher order spectral distortions as results of mode couplings as in the case with the usual y distortion.

6.3 First order Kompaneets terms

So far we have discussed the Thomson limit to ignore the nonlinear terms of f for simplicity. The above prescription itself can be powerful since it is applicable for the same class of collision process; however we should take into account not only the inhomogeneity but also the momentum transfer in the realistic application to the CMB. When we discussed the second order theory, the Kompaneets terms are comparable to the Thomson anisotropic parts; however, they are homogenous and do not contribute to the perturbation equation. The total average part is calculated by combining the result of the Thomson part with the SZ effects. If we look at the cubic order, the momentum transfer is expected to be written as products of the Compton y parameter and the first order anisotropies. In this case, the linear Kompaneets terms are non-negligible for the perturbation equations.

We now discuss the momentum transfer coming from $p(1+z)/m_e$ and T_e/m_e at cubic order. From (2.36) and (2.38), we have

$$(n_e \sigma_{\text{T}} a)^{-1} \left(\mathcal{C}_{\text{K},0}^{(0)}[f] + \mathcal{C}_{\text{K},0}^{(1)}[f] \right) = \frac{1}{m_e p^2} \frac{\partial}{\partial p} p^4 \left(T_e(1 + \Theta_{e0}) \frac{\partial f_0}{\partial p} + f_0 [1 + f_0] \right), \quad (6.17)$$

where $f_0 = f^{(0)} + f_0^{(1)}$, and we replace $T_e \rightarrow T_e(1 + \Theta_{e0})$ to include the electron temperature perturbation. The differentiated part can be calculated as

$$T_e(1 + \Theta_{e0}) \frac{\partial f_0}{\partial p} + f_0 [1 + f_0] = \left[T_e(1 + \Theta_{e0}) - T_0 e^{\tilde{\Theta}_0} \right] \frac{\partial f_0}{\partial p} - \frac{T_0 e^{\tilde{\Theta}_0}}{1 - p \frac{\partial \tilde{\Theta}_0}{\partial p}} \frac{\partial \tilde{\Theta}_0}{\partial p} p \frac{\partial f_0}{\partial p}. \quad (6.18)$$

Therefore, (6.17) yields

$$(n_e \sigma_{\text{T}} a)^{-1} \mathcal{C}_{\text{K},0}^{(1)}[f] \simeq \frac{1}{m_e p^2} \frac{\partial}{\partial p} p^4 \left[T_e(1 + \Theta_{e0}^{(1)}) - T_\gamma(1 + \Theta_0^{(1)}) \right] \frac{\partial f_0}{\partial p}. \quad (6.19)$$

The momentum independence of $\Theta^{(1)}$ is important for this expression, and we have non-trivial additional terms at higher order. Thus, the monopole component of the first order Kompaneets equation is obtained as follows ⁶:

⁶The terms proportional to v do not contain the zeroth order distribution functions so that they are the second order Kompaneets terms.

$$\begin{aligned}
(n_e \sigma_T a)^{-1} \mathcal{C}_{K,0}^{(1)}[f] &= \frac{1}{m_e p^2} \frac{\partial}{\partial p} p^4 \left[(T_e \Theta_{e0} - T_\gamma \Theta_0) \frac{\partial f^{(0)}}{\partial p} + (T_e - T_\gamma) \Theta \frac{\partial \mathcal{G}}{\partial p} \right] \\
&= \frac{T_e \Theta_{e0} - T_\gamma \Theta_0}{m_e} \mathcal{Y} + \frac{T_e - T_\gamma}{m_e} \Theta_0 \mathcal{K},
\end{aligned} \tag{6.20}$$

where we use

$$\frac{1}{p^2} \frac{\partial}{\partial p} p^4 \frac{\partial}{\partial p} = -3 \left(-p \frac{\partial}{\partial p} \right) + \left(-p \frac{\partial}{\partial p} \right)^2, \tag{6.21}$$

and

$$\frac{1}{p^2} \frac{\partial}{\partial p} p^4 \frac{\partial}{\partial p} \mathcal{G} = \mathcal{K}. \tag{6.22}$$

The other cubic order terms should be linear combinations of \mathcal{G} , \mathcal{Y} and \mathcal{K} as pointed above.

6.4 Linear Sunyaev-Zel'dovich effect

(6.20) has terms proportional to \mathcal{Y} and \mathcal{K} . The first term implies that there are additional sources for (3.33). Assuming that $T_e = T_\gamma$,

$$\frac{T_e \Theta_{e0} - T_\gamma \Theta_0}{m_e} = \frac{T_\gamma}{3m_e} S_{e\gamma}, \tag{6.23}$$

where we have defined the baryon isocurvature perturbation as

$$S_{e\gamma} = 3(\Theta_{e0} - \Theta_0) = \delta_e - \frac{3}{4} \delta_\gamma. \tag{6.24}$$

This implies that the yT cross correlation function does exist even for Gaussian perturbations suppose that there are baryon isocurvature perturbations and that they are cross correlated with the adiabatic ones. Physical implication of (6.23) is clear: the fluctuations of relative number density induce additional recoil effects. These terms may be crucial since $T_\gamma/m_e = \mathcal{O}(10^{-9})(1+z)$, which may be comparable to the acoustic source for $z \gtrsim \mathcal{O}(10^3)$.

On the other hand, for the adiabatic initial condition $\Theta_{e0} = \Theta_0$, one finds

$$n_e \sigma_T a \frac{T_e \Theta_{e0} - T_\gamma \Theta_0}{m_e} = \dot{y}_C \Theta_0. \tag{6.25}$$

This should be more important since y_C is recently estimated in [61], and the magnitude is expected to be 10^{-6} . Therefore, we roughly expect

$$y_0^{(3)} \sim \int d\eta \dot{y}_C \Theta_0 \sim y_C \Theta_0(z_c), \tag{6.26}$$

where z_c is the redshift when SZ effects occur. The cross correlation with the temperature can be given as

$$C_l^{y^{(3)}T} \sim 10^{-6} C_l^{TT}. \tag{6.27}$$

If we compare (6.27) to the non-Gaussianity origin $y^{(2)}T$ cross correlation, this corresponds to $f_{\text{NL}}^{\text{loc}} \sim 10$ [53], that is, we cannot ignore this contribution for the non-Gaussianity observation by using yT cross correlation; however, we also point out that the systematic errors for the PIXIE experiment is 10^3 times larger than the signal [62] and the problem is not so simple.

The second term in (6.20) also implies the other higher order SZ effects. As in the case with $y^{(3)}$, we estimate the higher order spectral distortion as

$$\kappa_0^{(3)} \sim y_{\text{C}} \Theta_0(z_c). \quad (6.28)$$

This implies that there are two types of linear SZ effects, and we can distinguish this higher order distortion from the former tSZ effect due to the momentum dependence. We should note that the anisotropies in the distortions are connected with the electron gas configurations, and it may be possible to have a 3D map of the linear perturbations by using the linear SZ effects.

6.5 Higher order spectral distortions and a residual distortion

Recently, non- μ and non- y type spectral distortions called *residual distortion* is proposed for the purpose of classifying the actual observational data of CMB intensity spectrum [63]. The authors introduced the n dimensional Euclidean space with n being a number of frequency channels and characterize the distortions by using linearly independent vectors in the space. The residual distortion is the fourth direction perpendicular to temperature shift, y distortion and μ distortion directions. Actually, there are $n - 3$ linearly independent directions for residual distortion, and our higher order momentum basis $\mathcal{Y}^{(n)}$ should be included in them. The residual distortion mainly well describes thermal history during μ - y transition period, which cannot be treated in our method due to the non-linearity of the distribution functions in Kompaneets terms as seen in (6.19). Full parametrization of the residual distortion in a systematic approach should be important for the future observational cosmology.

7 Summary

The second order temperature perturbations are momentum dependent in contrast to zeroth and first order. The momentum is usually integrated to obtain the second order brightness perturbations so that non-trivial configurations in the momentum spectrum has not been analyzed. In this paper, we explicitly wrote the second order Boltzmann equation for the Planck distribution function with a momentum dependent temperature and showed that such a dependence is separated into two “linearly independent” functions with corresponding parameters. One of them is understood as the fluctuations of the local blackbodies, and we derived the evolution equation for the acoustic temperature rise in a explicit way. Another is the form of well known y distortion which arise from Silk damping. We derived the exact evolution equation of the distortion and combine it with the homogenous component coming from the other thermal history. On the other hand, we also showed that the formation of the

spectral μ distortion is not understood in our framework. The μ is a result of the frequent momentum transfer, and the momentum independent ansatz does not work to explain the generation. In the last section, we also discussed the potential to extend our method to higher order. In our cases, the linearity of the distribution functions in the collision terms is crucial. As an example, we investigated the cubic order Boltzmann equations. We derived the cubic order Thomson terms and linear Kompaneets terms, and newly define the higher order y distortion to make the equations closed. We also showed that the mode coupling arises as in the case with y distortions, and found linear SZ effects. The above method has potential for applying to wider classes of non-equilibrium physics or non-linear problems. The basis functions may have the different forms depending on concrete collision terms; however several classes can be solved systematically as we have shown in this paper. For example, the Maxwell-Boltzmann distribution functions with several orders of the spectral distortions may be another window for the analysis of large scale structure, and Boltzmann equations for the massive neutrino might be solved in the same manner.

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A Multipole and harmonic expansion

In this paper, the multipole expansion of X is defined as

$$X(\hat{\mathbf{v}} \cdot \mathbf{n}) = \sum_l (-i)^l (2l+1) P_l(\hat{\mathbf{v}} \cdot \mathbf{n}) X_l, \quad (\text{A.1})$$

where $\hat{\mathbf{v}}$ is the direction of the baryon bulk velocity. On the other hand, the harmonic expansion is introduced as

$$X(\mathbf{n}) = \sum_{l,m} X_{lm} Y_{lm}(\mathbf{n}). \quad (\text{A.2})$$

The relation between these coefficients for $m=0$ can be expressed as

$$X_{l0} = \sqrt{4\pi(2l+1)} (-i)^l X_l. \quad (\text{A.3})$$

For example, expanding (3.20) and (3.21) in terms of multipoles, we can write the coefficients as follows:

$$f_l^{(1)} = \Theta_l^{(1)} \mathcal{G}, \quad (\text{A.4})$$

$$f_l^{(2)} = \left(\Theta_l^{(2)} + \frac{3}{2} [\Theta^2]_l \right) \mathcal{G} + \left(y_l + \frac{1}{2} [\Theta^2]_l \right) \mathcal{Y} \quad (\text{A.5})$$

Products of $(\mathbf{n} \cdot \mathbf{n}')$ can be separated by using the following formula:

$$\int \frac{d\mu'}{2} \int \frac{d\phi'}{2\pi} P_l(\mathbf{n} \cdot \mathbf{n}') = \int \frac{d\mu'}{2} P_l(\mu) P_l(\mu'), \quad (\text{A.6})$$

where $\mu(\mu')$ is the cosine between a z axis and $\mathbf{n}(\mathbf{n}')$.

B A translation of second order collision terms

While we reproduce the second order Boltzmann collision terms which was derived in a previous literature, we comment on the correspondence of the variables with the result in [2]. (4.42) in [2] is written by

$$\begin{aligned} \frac{1}{2} \mathcal{C}[f] (n_e \sigma_T)^{-1} = & \frac{1}{2} f_{00}^{(2)} - \frac{1}{4} \sum_{m=-2}^2 \frac{\sqrt{4\pi}}{5^{3/2}} f_{2m}^{(2)} Y_{2m} - \frac{1}{2} f^{(2)}(\mathbf{p}) \\ & + \delta_e^{(1)} \left[f_0^{(1)} + \frac{1}{2} f_2^{(1)} P_2(\hat{\mathbf{v}} \cdot \mathbf{n}) - f^{(1)} - p \frac{\partial f^{(0)}}{\partial p} (\mathbf{v} \cdot \mathbf{n}) \right] \\ & - \frac{1}{2} p \frac{\partial f^{(0)}}{\partial p} (\mathbf{v}^{(2)} \cdot \mathbf{n}) \\ & + (\mathbf{v} \cdot \mathbf{n}) \left[f^{(1)}(\mathbf{p}) - f_0^{(1)} - p \frac{\partial f_0^{(1)}}{\partial p} - f_2^{(1)} + \frac{1}{2} P_2(\hat{\mathbf{v}} \cdot \mathbf{n}) \left(f_2^{(1)} - p \frac{\partial f_2^{(1)}}{\partial p} \right) \right] \\ & + v \left[2f_1^{(1)} + p \frac{\partial f_1^{(1)}}{\partial p} + \frac{1}{5} P_2(\hat{\mathbf{v}} \cdot \mathbf{n}) \left(-f_1^{(1)}(p) + p \frac{\partial f_1^{(1)}}{\partial p} + 6f_3^{(1)} + \frac{3}{2} p \frac{\partial f_3^{(1)}}{\partial p} \right) \right] \\ & + (\mathbf{v} \cdot \mathbf{n})^2 \left[p \frac{\partial f^{(0)}}{\partial p} + \frac{11}{20} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} \right] + v^2 \left[p \frac{\partial f^{(0)}}{\partial p} + \frac{3}{20} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} \right] \\ & + \frac{1}{m_e p^2} \frac{\partial}{\partial p} \left[p^4 \left(T_e \frac{\partial f^{(0)}}{\partial p} + f^{(0)} (1 + f^{(0)}) \right) \right], \end{aligned} \quad (\text{B.1})$$

where we found p^2 in the denominator of the last line as also pointed out in [64]. In our notation, the time coordinate is conformal time η and we replace the functions as

$$\mathbf{p} \rightarrow \tilde{\mathbf{p}}, \quad (\text{B.2})$$

$$\frac{1}{2} \mathcal{C}[f] \rightarrow \mathcal{C}[f], \quad (\text{B.3})$$

$$\frac{1}{2} f^{(2)} \rightarrow f^{(2)}, \quad (\text{B.4})$$

$$\frac{1}{2} \mathbf{v}^{(2)} \rightarrow \mathbf{v}^{(2)} \quad (\text{B.5})$$

$$f_l \rightarrow (-i)^l f_l, \quad (\text{B.6})$$

$$f_{lm} \rightarrow (-i)^{-l} \sqrt{\frac{2l+1}{4\pi}} f_{lm}. \quad (\text{B.7})$$

C A treatment to convolutions

The linear perturbations are simply proportional to the primordial perturbations in Fourier space. Therefore, the Boltzmann equations are also the equations for the transfer functions simultaneously. This is not the case at second order since we have convolutions, namely, the Fourier momentum integrals which include the primordial curvature perturbations. Let us write the curvature perturbations in the convolution explicitly as follows:

$$(XY)_{\mathbf{k}} \equiv \int \frac{d^3q}{(2\pi)^3} X_{\mathbf{q}} Y_{\mathbf{k}-\mathbf{q}} \mathcal{R}_{\mathbf{q}} \mathcal{R}_{\mathbf{k}-\mathbf{q}}, \quad (\text{C.1})$$

where let $X_{\mathbf{q}}$ and $Y_{\mathbf{k}-\mathbf{q}}$ be the transfer functions for the linear perturbations. The ensemble average with $\mathcal{R}_{\mathbf{k}'}$ is then given as

$$\langle (XY)_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \int \frac{d^3q}{(2\pi)^3} X_{\mathbf{q}} Y_{\mathbf{k}-\mathbf{q}} B_{\mathcal{R}}(q, |\mathbf{k} - \mathbf{q}|, k'), \quad (\text{C.2})$$

where $B_{\mathcal{R}}$ is the shape function of the primordial bispectrum. Let us consider the case X and Y are significant for $k \ll q$, and let us integrate q in advance. Then, we can approximate the above equation as

$$\langle (XY)_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle \simeq (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \int \frac{d^3q}{(2\pi)^3} X_q Y_q B_{\mathcal{R}}(q, q, k'). \quad (\text{C.3})$$

Suppose that the bispectrum is local-type, one can simplify this further and obtain

$$\langle (XY)_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') P_{\mathcal{R}}(k') \left(-\frac{12}{5} f_{\text{NL}}^{\text{loc}} \right) \int \frac{dq}{q} \mathcal{P}_{\mathcal{R}}(q) X_q Y_q. \quad (\text{C.4})$$

This expression tells us that it is equivalent to replace the convolution as

$$(XY)_{\mathbf{k}} \sim \int \frac{dq}{q} \mathcal{P}_{\mathcal{R}}(q) X_q Y_q \mathcal{R}_{\mathbf{k}}^{(2)}, \quad (\text{C.5})$$

where we have defined the “second order curvature perturbation” to satisfy

$$\langle \mathcal{R}_{\mathbf{k}}^{(2)} \mathcal{R}_{\mathbf{k}'} \rangle \sim (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \left(-\frac{12}{5} f_{\text{NL}}^{\text{loc}} \right) P_{\mathcal{R}}(k), \quad (\text{C.6})$$

$$\langle \mathcal{R}_{\mathbf{k}}^{(2)} \mathcal{R}_{\mathbf{k}'}^{(2)} \rangle \sim (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') 4\tau_{\text{NL}}^{\text{loc}} P_{\mathcal{R}}(k). \quad (\text{C.7})$$

References

- [1] P. A. R. Ade et al. Planck 2015 results. XVII. Constraints on primordial non-Gaussianity. *Astron. Astrophys.*, 594:A17, 2016.
- [2] Nicola Bartolo, Sabino Matarrese, and Antonio Riotto. CMB Anisotropies at Second Order I. *JCAP*, 0606:024, 2006.
- [3] Cyril Pitrou. Gauge invariant Boltzmann equation and the fluid limit. *Class. Quant. Grav.*, 24:6127–6158, 2007.

- [4] Cyril Pitrou. The radiative transfer for polarized radiation at second order in cosmological perturbations. *Gen. Rel. Grav.*, 41:2587–2595, 2009.
- [5] M. Beneke and C. Fidler. Boltzmann hierarchy for the cosmic microwave background at second order including photon polarization. *Phys. Rev.*, D82:063509, 2010.
- [6] Atsushi Naruko, Cyril Pitrou, Kazuya Koyama, and Misao Sasaki. Second-order Boltzmann equation: gauge dependence and gauge invariance. *Class. Quant. Grav.*, 30:165008, 2013.
- [7] N. Bartolo, Sabino Matarrese, and A. Riotto. Evolution of second - order cosmological perturbations and non-Gaussianity. *JCAP*, 0401:003, 2004.
- [8] Nicola Bartolo, Sabino Matarrese, and Antonio Riotto. Gauge-invariant temperature anisotropies and primordial non-Gaussianity. *Phys. Rev. Lett.*, 93:231301, 2004.
- [9] N. Bartolo, S. Matarrese, and A. Riotto. Non-Gaussianity in the Cosmic Microwave Background Anisotropies at Recombination in the Squeezed limit. *JCAP*, 1202:017, 2012.
- [10] Leonardo Senatore, Svetlin Tassev, and Matias Zaldarriaga. Non-Gaussianities from Perturbing Recombination. *JCAP*, 0909:038, 2009.
- [11] Rishi Khatri and Benjamin D. Wandelt. Crinkles in the last scattering surface: Non-Gaussianity from inhomogeneous recombination. *Phys. Rev.*, D79:023501, 2009.
- [12] Daisuke Nitta, Eiichiro Komatsu, Nicola Bartolo, Sabino Matarrese, and Antonio Riotto. CMB anisotropies at second order III: bispectrum from products of the first-order perturbations. *JCAP*, 0905:014, 2009.
- [13] Paolo Creminelli and Matias Zaldarriaga. CMB 3-point functions generated by non-linearities at recombination. *Phys. Rev.*, D70:083532, 2004.
- [14] Lotfi Boubekur, Paolo Creminelli, Guido D’Amico, Jorge Norena, and Filippo Vernizzi. Sachs-Wolfe at second order: the CMB bispectrum on large angular scales. *JCAP*, 0908:029, 2009.
- [15] Antony Lewis. The full squeezed CMB bispectrum from inflation. *JCAP*, 1206:023, 2012.
- [16] Paolo Creminelli, Cyril Pitrou, and Filippo Vernizzi. The CMB bispectrum in the squeezed limit. *JCAP*, 1111:025, 2011.
- [17] Christian Fidler, Kazuya Koyama, and Guido W. Pettinari. A new line-of-sight approach to the non-linear Cosmic Microwave Background. *JCAP*, 1504(04):037, 2015.
- [18] Cyril Pitrou, Jean-Philippe Uzan, and Francis Bernardeau. The cosmic microwave background bispectrum from the non-linear evolution of the cosmological perturbations. *JCAP*, 1007:003, 2010.
- [19] S. C. Su, Eugene A. Lim, and E. P. S. Shellard. CMB Bispectrum from Non-linear Effects during Recombination. 2012.
- [20] Zhiqi Huang and Filippo Vernizzi. Cosmic Microwave Background Bispectrum from Recombination. *Phys. Rev. Lett.*, 110(10):101303, 2013.
- [21] Guido W. Pettinari, Christian Fidler, Robert Crittenden, Kazuya Koyama, and David Wands. The intrinsic bispectrum of the Cosmic Microwave Background. *JCAP*, 1304:003, 2013.

- [22] Zhiqi Huang and Filippo Vernizzi. The full CMB temperature bispectrum from single-field inflation. *Phys. Rev.*, D89(2):021302, 2014.
- [23] Ryo Saito, Atsushi Naruko, Takashi Hiramatsu, and Misao Sasaki. Geodesic *curve-of-sight* formulae for the cosmic microwave background: a unified treatment of redshift, time delay, and lensing. *JCAP*, 1410(10):051, 2014.
- [24] Cyril Pitrou, Francis Bernardeau, and Jean-Philippe Uzan. The y-sky: diffuse spectral distortions of the cosmic microwave background. *JCAP*, 1007:019, 2010.
- [25] Sebastien Renaux-Petel, Christian Fidler, Cyril Pitrou, and Guido W. Pettinari. Spectral distortions in the cosmic microwave background polarization. *JCAP*, 1403:033, 2014.
- [26] Ya. B. Zeldovich and R. A. Sunyaev. The Interaction of Matter and Radiation in a Hot-Model Universe. *Astrophys. Space Sci.*, 4:301–316, 1969.
- [27] R. A. Sunyaev and Ya. B. Zeldovich. The Interaction of matter and radiation in the hot model of the universe. *Astrophys. Space Sci.*, 7:20–30, 1970.
- [28] N. Aghanim et al. Planck 2015 results. XXII. A map of the thermal Sunyaev-Zeldovich effect. *Astron. Astrophys.*, 594:A22, 2016.
- [29] W. Hu and J. Silk. Thermalization constraints and spectral distortions for massive unstable relic particles. *Phys. Rev. Lett.*, 70:2661–2664, 1993.
- [30] Patrick McDonald, Robert J. Scherrer, and Terry P. Walker. Cosmic microwave background constraint on residual annihilations of relic particles. *Phys. Rev.*, D63:023001, 2001.
- [31] J. Chluba. Could the Cosmological Recombination Spectrum Help Us Understand Annihilating Dark Matter? *Mon. Not. Roy. Astron. Soc.*, 402:1195, 2010.
- [32] B. J. Carr, Kazunori Kohri, Yuuiti Sendouda, and Jun’ichi Yokoyama. New cosmological constraints on primordial black holes. *Phys. Rev.*, D81:104019, 2010.
- [33] R. A. Sunyaev and Ya. B. Zeldovich. Small scale fluctuations of relic radiation. *Astrophys. Space Sci.*, 7:3–19, 1970.
- [34] Wayne Hu, Douglas Scott, and Joseph Silk. Power spectrum constraints from spectral distortions in the cosmic microwave background. *Astrophys. J.*, 430:L5–L8, 1994.
- [35] J. Chluba and R. A. Sunyaev. The evolution of CMB spectral distortions in the early Universe. *Mon. Not. Roy. Astron. Soc.*, 419:1294–1314, 2012.
- [36] J. D. Barrow and P. Coles. Primordial density fluctuations and the microwave background spectrum. *mnras*, 248:52–57, January 1991.
- [37] Jens Chluba, Rishi Khatri, and Rashid A. Sunyaev. CMB at 2x2 order: The dissipation of primordial acoustic waves and the observable part of the associated energy release. *Mon. Not. Roy. Astron. Soc.*, 425:1129–1169, 2012.
- [38] Jens Chluba, Adrienne L. Erickcek, and Ido Ben-Dayan. Probing the inflaton: Small-scale power spectrum constraints from measurements of the CMB energy spectrum. *Astrophys. J.*, 758:76, 2012.
- [39] Rishi Khatri, Rashid A. Sunyaev, and Jens Chluba. Mixing of blackbodies: entropy production and dissipation of sound waves in the early Universe. *Astron. Astrophys.*, 543:A136, 2012.

- [40] Rishi Khatri and Rashid A. Sunyaev. Beyond y and μ : the shape of the CMB spectral distortions in the intermediate epoch, $1.5 \times 10^4 < z < 2 \times 10^5$. *JCAP*, 1209:016, 2012.
- [41] Rishi Khatri and Rashid A. Sunyaev. Forecasts for CMB μ and i -type spectral distortion constraints on the primordial power spectrum on scales $8 \lesssim k \lesssim 10^4 \text{ Mpc}^{-1}$ with the future Pixie-like experiments. *JCAP*, 1306:026, 2013.
- [42] Sebastien Clesse, Bjorn Garbrecht, and Yi Zhu. Testing Inflation and Curvaton Scenarios with CMB Distortions. *JCAP*, 1410(10):046, 2014.
- [43] Atsuhisa Ota, Tomo Takahashi, Hiroyuki Tashiro, and Masahide Yamaguchi. CMB μ distortion from primordial gravitational waves. *JCAP*, 1410(10):029, 2014.
- [44] Jens Chluba, Liang Dai, Daniel Grin, Mustafa Amin, and Marc Kamionkowski. Spectral distortions from the dissipation of tensor perturbations. *Mon. Not. Roy. Astron. Soc.*, 446:2871–2886, 2015.
- [45] Enrico Pajer and Matias Zaldarriaga. A New Window on Primordial non-Gaussianity. *Phys. Rev. Lett.*, 109:021302, 2012.
- [46] Enrico Pajer and Matias Zaldarriaga. A hydrodynamical approach to CMB μ -distortion from primordial perturbations. *JCAP*, 1302:036, 2013.
- [47] Jonathan Ganc and Eiichiro Komatsu. Scale-dependent bias of galaxies and μ -type distortion of the cosmic microwave background spectrum from single-field inflation with a modified initial state. *Phys. Rev.*, D86:023518, 2012.
- [48] Jonathan Ganc and Martin S. Sloth. Probing correlations of early magnetic fields using μ -distortion. *JCAP*, 1408:018, 2014.
- [49] Atsuhisa Ota, Toyokazu Sekiguchi, Yuichiro Tada, and Shuichiro Yokoyama. Anisotropic CMB distortions from non-Gaussian isocurvature perturbations. *JCAP*, 1503(03):013, 2015.
- [50] Atsushi Naruko, Atsuhisa Ota, and Masahide Yamaguchi. Probing small-scale non-Gaussianity from anisotropies in acoustic reheating. *JCAP*, 1505(05):049, 2015.
- [51] Atsuhisa Ota. Cosmological constraints from μE cross-correlations. 2016.
- [52] Razieh Emami, Emanuela Dimastrogiovanni, Jens Chluba, and Marc Kamionkowski. Probing the scale dependence of non-Gaussianity with spectral distortions of the cosmic microwave background. *Phys. Rev.*, D91(12):123531, 2015.
- [53] Jens Chluba, Emanuela Dimastrogiovanni, Mustafa A. Amin, and Marc Kamionkowski. Evolution of CMB spectral distortion anisotropies and tests of primordial non-Gaussianity. 2016.
- [54] Scott Dodelson and Jay M. Jubas. Reionization and its imprint on the cosmic microwave background. *Astrophys. J.*, 439:503–516, 1995.
- [55] Albert Stebbins. CMB Spectral Distortions from the Scattering of Temperature Anisotropies. *Submitted to: Phys. Rev. D*, 2007.
- [56] Cyril Pitrou and Albert Stebbins. Parameterization of temperature and spectral distortions in future CMB experiments. *Gen. Rel. Grav.*, 46(11):1806, 2014.
- [57] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. 1995.

- [58] Donghui Jeong, Josef Pradler, Jens Chluba, and Marc Kamionkowski. Silk damping at a redshift of a billion: a new limit on small-scale adiabatic perturbations. *Phys. Rev. Lett.*, 113:061301, 2014.
- [59] Tomohiro Nakama, Teruaki Suyama, and Jun’ichi Yokoyama. Reheating the Universe Once More: The Dissipation of Acoustic Waves as a Novel Probe of Primordial Inhomogeneities on Even Smaller Scales. *Phys. Rev. Lett.*, 113:061302, 2014.
- [60] Chung-Pei Ma and Edmund Bertschinger. Cosmological perturbation theory in the synchronous and conformal Newtonian gauges. *Astrophys. J.*, 455:7–25, 1995.
- [61] J. Colin Hill, Nick Battaglia, Jens Chluba, Simone Ferraro, Emmanuel Schaan, and David N. Spergel. Taking the Universes Temperature with Spectral Distortions of the Cosmic Microwave Background. *Phys. Rev. Lett.*, 115(26):261301, 2015.
- [62] A. Kogut et al. The Primordial Inflation Explorer (PIXIE): A Nulling Polarimeter for Cosmic Microwave Background Observations. *JCAP*, 1107:025, 2011.
- [63] Jens Chluba and Donghui Jeong. Teasing bits of information out of the CMB energy spectrum. *Mon. Not. Roy. Astron. Soc.*, 438(3):2065–2082, 2014.
- [64] Leonardo Senatore, Svetlin Tassev, and Matias Zaldarriaga. Cosmological Perturbations at Second Order and Recombination Perturbed. *JCAP*, 0908:031, 2009.